LECTURE 5

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Contents

Notation

1. Lecture 5, a naive explanation of the low and high entropy method

1.1. **Prelude.** In this lecture, we present the key idea of the EKL paper [EKL06]: the high and low entropy method. We are going to make some (too strong) assumptions under which the idea of these methods shall be explained.

The key are unipotent matrices and their interplay with diagonal matrices. Unipotent matrices could be sources of being unbounded. For instance, if Γ is a discrete subgroup of $G = \mathbf{SL}_2(\mathbb{R})$ (or any other semisimple linear Lie group) that contains some non-trivial unipotent matrix, then G/Γ is non-compact.

1.2. **Notation.**

$$\mathrm{A} = \left\{ \begin{bmatrix} e^{t_1} & 0 & 0 \\ 0 & e^{t_2} & 0 \\ 0 & 0 & e^{t_3} \end{bmatrix} \middle| \sum t_i = 0 \right\}$$

$$\mathrm{A}^+ = \left\{ \begin{bmatrix} e^{t_1} & 0 & 0 \\ 0 & e^{t_2} & 0 \\ 0 & 0 & e^{t_3} \end{bmatrix} \in \mathrm{A} \middle| t_1, t_2 > 0 \right\}.$$

For $i \neq j$, let E_{ij} be the matrix whose (i,j)-entry is one and is zero elsewhere. Let $\mathbf{u}_{ij}(r) := \mathrm{I}_3 + rE_{ij}$ and $\mathrm{U}_{ij} := \{\mathbf{u}_{ij}(r),\ r \in \mathbb{R}\}$. For instance,

$$\mathrm{U}_{12} := \left\{ \mathbf{u}_{12}(s) = \begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| s \in \mathbb{R} \right\}, \quad \mathrm{U}_{13} := \left\{ \mathbf{u}_{13}(s) = \begin{bmatrix} 1 & 0 & s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| s \in \mathbb{R} \right\},$$

$$\mathrm{U}_{23} := \left\{ \mathbf{u}_{23}(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \middle| s \in \mathbb{R} \right\}.$$

Also for (i,j,k), an ordering of $\{1,2,3\}$, let $\mathrm{U}_{ijk} := \mathrm{U}_{ij}\mathrm{U}_{ik}\mathrm{U}_{jk}$, $\mathrm{U}_{ij,ik} := \mathrm{U}_{ij}\mathrm{U}_{ik}$ and $\mathrm{U}_{ik,jk} := \mathrm{U}_{ik}\mathrm{U}_{jk}$. These are subgroups. For instance:

$$\mathrm{U}_{123} := \left\{ \begin{bmatrix} 1 & r & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \middle| r,s,t \in \mathbb{R} \right\}, \quad \mathrm{U}_{12,13} := \left\{ \begin{bmatrix} 1 & r & s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| r,s \in \mathbb{R} \right\}.$$

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It is also useful to note that the centralizer of U_{13} is

$$\mathrm{Z}_{\mathbf{SL}_3}(\mathrm{U}_{13}) = \left\{ \begin{bmatrix} t & u_{12} & u_{13} \\ 0 & t^{-2} & u_{23} \\ 0 & 0 & t \end{bmatrix} \right\}.$$

1.3. **Recurrence leaf.** [1]

Recall that for $(\alpha, \beta) \in \mathbb{R}^2$, we let

$$\Lambda_{\alpha,\beta} := \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} .\mathbb{Z}^3.$$

For $E \subset [0,1)^2$,

$$\mathscr{C}_E := \left\{ x \;\middle|\; x = \lim a_n.\Lambda_{\alpha,\beta},\ \exists\,(\alpha,\beta) \in E \text{ and divergent } (a_n) \subset \mathrm{A}^+ \right\}.$$

For $x \in \mathrm{X}_3$ and an ordering (ijk) of $\{1,2,3\}$, let

$$\mathscr{C}_x^{ij} := \{u \in \mathrm{U}_{ij} \mid u.x \in \mathscr{C}_E\}$$
$$\mathscr{C}_x^{ij,ik} := \{u \in \mathrm{U}_{ij,ik} \mid u.x \in \mathscr{C}_E\},\ \mathscr{C}_x^{ik,jk} := \{u \in \mathrm{U}_{ik,jk} \mid u.x \in \mathscr{C}_E\}$$
$$\mathscr{C}_x^{ijk} := \{u \in \mathrm{U}_{ijk} \mid u.x \in \mathscr{C}_E\}$$

Lemma 1.1. *These sets satisfy certain formal properties such as*

1. *for* $u \in \mathrm{U}_{ij}$, $\mathscr{C}_{u.x}^{ij} \cdot u = \mathscr{C}_x^{ij}$;
2. *for* $a = \mathrm{diag}(a_1, a_2, a_3) \in \mathrm{A}$, $\mathscr{C}_{a.x}^{ij} = a\mathscr{C}_x^{ij}a^{-1}$.

Now fix some $E \subset [0,1)^2$ such that $\mathrm{A}^+.\Lambda_E$ is contained in some compact subset of X_3. From now on, we make the following (too strong) assumptions[2]:

Assumption 1.2.
- *The map $x \mapsto \mathscr{C}_x^\star$ is continuous from $\mathscr{C}_E$ to the set of closed subsets*[3] *of some $\mathrm{U}_\star$ for any $\star = (ij)$, (ij,ik), (ik,jk) or (ijk);*
- *For every ordering (ijk) of $\{1,2,3\}$, $\mathscr{C}_x^{ijk}$ being infinite for every $x \in \mathscr{C}_E$ is equivalent to $\mathscr{C}_x^{kji}$ being infinite for every $x \in \mathscr{C}_E$.*
- *for every $i \neq j$, we have the following dichotomy: either $\mathscr{C}_x^{ij}$ is a singleton $\{\mathrm{I}_3\}$ for every $x \in \mathscr{C}_E$ or $\mathscr{C}_x^{ij}$ is infinite for every $\mathscr{C}_E$;*
- *there exists $i \neq j$ such that $\mathscr{C}_x^{ij}$ is infinite*[4] *for every $x \in \mathscr{C}_E$.*

Corollary 1.3. *Under the above assumptions, if $\mathscr{C}_x^{ij}$ is infinite for every $x \in \mathscr{C}_E$, then $\mathscr{C}_x^{ij}$ contains arbitrarily small non-identity elements for every $x \in \mathscr{C}_E$;*

Proof. If for some $x \in \mathscr{C}_E$, one can find $\rho > 0$ with $\mathbf{u}_{ij}((-\rho,\rho)) \cap \mathscr{C}_x^{ij} = \{\mathrm{I}_3\}$, then

$$\mathbf{u}_{ij}((-a_i a_j^{-1}\rho, a_i a_j^{-1}\rho)) \cap \mathscr{C}_x^{ij} = \{\mathrm{I}_3\}, \quad \forall\, a = \mathrm{diag}(a_1,a_2,a_3) \in \mathrm{A}.$$

Choose $a(n) = \mathrm{diag}(a(n)_1, a(n)_2, a(n)_3)$ such that $a(n)_i/a(n)_j \to +\infty$. And let y be any limit point of $a(n).x$. Then by continuity, $\mathscr{C}_y^{ij} = \{0\}$. This is a contradiction. □

From now on assume E is non-empty and $\mathscr{C}_E$ is compact[5]. We would like to derive a contradiction. Let us actually make a statement in case it seems too vague to you.

Theorem 1.4. *Let $\mathscr{C}_E$ (the subscript E means nothing here) be an* A*-invariant compact subset of* X_3 *satisfying Assumption 1.2. Then $\mathscr{C}_E$ is empty.*

Anticipating the proof, we shall exhibit a $\mathrm{U}_{ij}^+ := \mathbf{u}_{ij}(\mathbb{R}_{\geq 0})$ or $\mathrm{U}_{ij}^- := \mathbf{u}_{ij}(\mathbb{R}_{\leq 0})$ orbit inside $\mathscr{C}_E$ for some $i \neq j$. But this would contradict against the following:

Lemma 1.5. *For each $i \neq j$ and $\star = +, -$, every orbit of the semigroup $\mathrm{A} \cdot \mathrm{U}_{ij}^\star$ on X_3 is unbounded.*

[1]Maybe the correct name should be recurrence set on leaves?

[2]The key assumption is the first one on continuity.

[3]A sequence of closed subsets (E_n) of $\mathbb{R}^n$ converges to E iff for every bounded open subset $O \subset \mathbb{R}^n$ the Hausdorff distance between $E_n \cap O$ and $E \cap O$ decreases to zero.

[4]One expects that this is likely to hold if $\dim E > 0$

[5]Recall that for (α,β) that fails Littlewood, we have that $\mathscr{C}_{(\alpha,\beta)}$ is compact

Proof. Without loss of generality assume $(i,j) = (2,3)$ and $\star = +$.

Take $\Lambda \in \mathrm{X}_3$ and $\mathbf{v} = (v_1, v_2, v_3) \in \Lambda$ with $v_3 < 0$ (every lattice would contain such a vector). By choosing suitable r, the lattice

$$u.\Lambda := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{bmatrix} .\Lambda$$

contains some vector $\mathbf{w} = (w_1, 0, w_2)$. Then one can find $a_n \in \mathrm{A}$ such that $a_n.\mathbf{w} \to \mathbf{0}$. By continuity of the systole function, $\{a_n u.\Lambda\}$ is unbounded in X_3. $\square$

1.4. **Product structure.** Roughly speaking, the lemma below says that "Recurrence leaf in the central direction is unchanged along unstable leaves".

Lemma 1.6. *Take $x \in \mathscr{C}_E$ and $u \in \mathrm{U}_{12}, v \in \mathrm{U}_{13,23}$ such that $y := uv.x \in \mathscr{C}_E$, then*

$$u \cdot \mathscr{C}^{12}_y = \mathscr{C}^{12}_x.$$

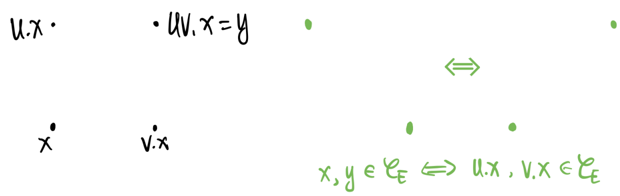
Proof. Take a sequence $(a_n) \subset \mathrm{A}$ such that conjugating by a_n contracts $\mathrm{U}_{13,23}$ and a_n commutes with U_{12} (e.g., take $a_n := \mathrm{diag}(n^{-1}, n^{-1}, n^2)$). Passing to a subsequence, assume that $a_n.x$ converges to $x_\infty \in \mathscr{C}_E$. By continuity of $\mathscr{C}^{12}_\bullet$,

$$\mathscr{C}^{12}_x = \mathscr{C}^{12}_{a_n.x} \to \mathscr{C}^{12}_{x_\infty} = u \cdot \mathscr{C}^{12}_{u.x_\infty} \leftarrow u \cdot \mathscr{C}^{12}_{a_n uv.x} = u \cdot \mathscr{C}^{12}_y.$$

$\square$

Corollary 1.7. *The product map $(g,h) \mapsto g \cdot h$ induces a bijection $\mathscr{C}^{12}_x \times \mathscr{C}^{13,23}_x \cong \mathscr{C}^{123}_x$ for every $x \in \mathscr{C}_E$.*

By similar arguments, $\mathscr{C}^{23}_x \times \mathscr{C}^{12,13}_x \cong \mathscr{C}^{123}_x$, $\mathscr{C}^{12}_x \times \mathscr{C}^{13}_x \cong \mathscr{C}^{12,13}_x$.... Soon we will see that these different decomposition of $\mathscr{C}^{123}_x$ lead to additional invariance.



Proof. Let $u \in \mathrm{U}_{12}, v \in \mathrm{U}_{13,23}$ and $x \in \mathscr{C}_E$. We need to show that

$$uv.x \in \mathscr{C}_E \iff u.x,\, v.x \in \mathscr{C}_E.$$

First we do the "$\Longrightarrow$" direction. Indeed, by Lemma 1.6,

$$uv.x \in \mathscr{C}_E \implies \mathrm{id} \in \mathscr{C}^{12}_{uv.x} = u^{-1} \cdot \mathscr{C}^{12}_x \implies u \in \mathscr{C}^{12}_x \implies u.x \in \mathscr{C}_E.$$

Similarly,

$$x \in \mathscr{C}_E \implies \mathrm{id} \in \mathscr{C}^{12}_x = u \cdot \mathscr{C}^{12}_{uv.x} \implies u^{-1} \in \mathscr{C}^{12}_{uv.x} \implies v.x \in \mathscr{C}_E.$$

For the reverse implication "$\Longleftarrow$", by Lemma 1.6,

$$x \in \mathscr{C}_E, v.x \in \mathscr{C}_E \implies \mathscr{C}^{12}_x = \mathscr{C}^{12}_{v.x}.$$

Therefore,

$$u.x \in \mathscr{C}_E \implies u \in \mathscr{C}^{12}_x = \mathscr{C}^{12}_{v.x} \implies uv.x \in \mathscr{C}_E.$$

$\square$

1.5. **Product structure vs. non-commutativity of the Heisenberg group.** Let us calculate the commutator $[u,v]$ for $u \in \mathrm{U}_{12}$ and $v \in \mathrm{U}_{23}$:

$$\begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & st \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

Namely,

$$\mathbf{u}_{12}(s)\mathbf{u}_{23}(t)\mathbf{u}_{12}(s)^{-1}\mathbf{u}_{23}(t)^{-1} = \mathbf{u}_{13}(st).$$

Lemma 1.8. *Take $x \in \mathscr{C}_E$. Assume that both $\mathscr{C}^{12}_x$ and $\mathscr{C}^{23}_x$ contain non-identity elements arbitrarily close to the identity, then $\mathscr{C}^{13}_x$ contains $\mathbf{u}_{13}(\mathbb{R}_{\geq 0})$ or $\mathbf{u}_{13}(\mathbb{R}_{\leq 0})$.*

Proof. Take non-zero $s_n \to 0$ and $t_n \to 0$ such that $\mathbf{u}_{12}(s_n) \in \mathscr{C}_x^{12}$ and $\mathbf{u}_{23}(t_n) \in \mathscr{C}_x^{23}$. Without loss of generality assume $s_n, t_n > 0$.

We are going to show that $\mathscr{C}_x^{13}$, when identified with a subset of $\mathbb{R}$, is invariant under addition by $s_n t_n$. Namely, taking $z \in \mathrm{U}_{13}$ with $z.x \in \mathscr{C}_E$, we need to show that $\mathbf{u}_{13}(s_n t_n) z.x \in \mathscr{C}_E$. Once this is done, a continuity argument shows that $\mathbb{R}_{\geq 0} + \mathscr{C}_x^{13} \subset \mathscr{C}_x^{13}$. In particular, $\mathscr{C}_x^{13}$ contains $\mathbb{R}_{\geq 0}$.

Note that $x \in \mathscr{C}_E$. By Corollary 1.7 (applied to $\mathrm{U}_{23} \times \mathrm{U}_{13}$),

$$\mathbf{u}_{23}(t_n).x,\ z.x \in \mathscr{C}_E \implies \mathbf{u}_{23}(t_n) z.x \in \mathscr{C}_E.$$

By Corollary 1.7 again (applied to $\mathrm{U}_{12} \times \mathrm{U}_{13,23}$),

$$\mathbf{u}_{12}(s_n).x,\ \mathbf{u}_{23}(t_n) z.x \in \mathscr{C}_E \implies \mathbf{u}_{12}(s_n)\mathbf{u}_{23}(t_n) z.x \in \mathscr{C}_E.$$

By Equa.(1), this is equivalent to

$$\mathbf{u}_{13}(s_n t_n).(\mathbf{u}_{23}(t_n)\mathbf{u}_{12}(s_n) z.x) = (\mathbf{u}_{23}(t_n)) \cdot (\mathbf{u}_{13}(s_n t_n)\mathbf{u}_{12}(s_n)).(z.x) \in \mathscr{C}_E.$$

By Corollary 1.7 and $z.x \in \mathscr{C}_E$,

$$\begin{aligned} & (\mathbf{u}_{23}(t_n)) \cdot (\mathbf{u}_{13}(s_n t_n)\mathbf{u}_{12}(s_n)).(z.x) \in \mathscr{C}_E \\ (\mathrm{U}_{23} \times \mathrm{U}_{12,13}) \quad & \implies \mathbf{u}_{13}(s_n t_n)\mathbf{u}_{23}(t_n).(z.x) \in \mathscr{C}_E \\ (\mathrm{U}_{13} \times \mathrm{U}_{12}) \quad & \implies \mathbf{u}_{13}(s_n t_n).(z.x) \in \mathscr{C}_E. \end{aligned}$$

So we are done. □

1.6. **Conclusion of the high entropy method.**

Lemma 1.9. *For every $x \in \mathscr{C}_E$, at most one of $\mathscr{C}_x^{12}, \mathscr{C}_x^{13}$ and $\mathscr{C}_x^{23}$ is infinite.*

Similarly at most one of $\mathscr{C}_x^{21}, \mathscr{C}_x^{23}$ and $\mathscr{C}_x^{13}$ is infinite.

There are essentially two cases to consider.

1.6.1. *Case I.* Assume that $\mathscr{C}_x^{12}$ and $\mathscr{C}_x^{23}$ are infinite. By Corollary 1.3 (2), $\mathscr{C}_x^{12}$ and $\mathscr{C}_x^{23}$ contains non-identity elements arbitrarily close to id. By Lemma 1.8, we may assume $\mathbf{u}_{13}(\mathbb{R}_{\geq 0})$ (the other case is similar) belongs to $\mathscr{C}_x^{13}$. So we have

$$\left\{ \begin{bmatrix} e^{t_1} & 0 & r_2 \\ 0 & e^{t_2} & 0 \\ 0 & 0 & e^{t_3} \end{bmatrix} \middle| \sum t_i = 0, r_2 \geq 0 \right\}.x \subset \mathscr{C}_E \text{ is bounded,} \tag{2}$$

which is impossible by Lemma 1.5.

1.6.2. *Case II.* Assume that $\mathscr{C}_x^{12}$ and $\mathscr{C}_x^{13}$ are infinite but $\mathscr{C}_x^{23}$ is finite for every x. By part (2) of the Assumption 1.2, $\mathscr{C}_x^{321}$ is infinite and hence by product structure at least one of $\mathscr{C}_x^{21}, \mathscr{C}_x^{31}, \mathscr{C}_x^{32}$ is infinite. Then similar arguments as in case I would lead to a contradiction against Lemma 1.5.

1.6.3. *One can avoid the use of the assumption here...* **We did not do this in the class. One can skip ahead to the Lemma below.**

We claim that at least one of $\mathscr{C}_x^{21}, \mathscr{C}_x^{31}, \mathscr{C}_x^{32}$ is infinite holds without invoking the part (2) of the assumption.

Now assume they are all finite. For $\eta > 0$, let[6]

$$H_\eta := \left\{ \begin{bmatrix} e^{t_1} & r_1 & r_2 \\ 0 & e^{t_2} & r_3 \\ 0 & 0 & e^{-t_1-t_2} \end{bmatrix} \middle| \left|t_1\right|, \left|t_2\right|, \left|r_1\right|, \left|r_2\right|, \left|r_3\right| < \eta \right\}$$

and

$$\theta_t := \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^t \end{bmatrix}$$

Then by Assumption 1.2, there exists $\delta_1, \eta_1 > 0$ small enough such that for every $x, x' \in \mathscr{C}_E$ with $d(x, x') < \delta_1 \implies x' \in H(\eta_1).x$.

[6] one can also impose $r_3 = 0$

Indeed, if this were not the case, by the “exponential blow-up” (see Lecture 4), we can construct $z \neq z' \in \mathscr{C}_E$ such that

$$z' = \begin{bmatrix} 1 & 0 & 0 \\ r_1 & 1 & 0 \\ r_2 & r_3 & 1 \end{bmatrix} .z$$

with r_1, r_2, r_3 arbitrarily close to 0. By Corollary 1.7 together with our assumption that these leaves are finite, this would contradict against Assumption 1.2.

Now, we can cover $\mathscr{C}_E$ by finitely many $\{H(\eta_1).x_i,\ i = 1, .., l\}$. Choose $t_n \to +\infty$ such that $z_i := \lim \theta_{t_n}.x_i$ exists for every i. Then[7]

$$\mathscr{C}_E = \theta_{t_n}.\mathscr{C}_E \subset \bigcup \theta_{t_n}.H(\eta_1).x_i \to \bigcup \mathrm{A}.z_i \subset \mathscr{C}_E.$$

Therefore, $\mathscr{C}_E$ is a finite union of A-orbits. So each of them is compact. This contradicts against our assumption[8].

Corollary 1.10. *If $\mathscr{C}_x^{23}$ is infinite for every $x \in \mathscr{C}_E$, then $\mathscr{C}_x^{21}$, $\mathscr{C}_x^{31}$, $\mathscr{C}_x^{12}$ and $\mathscr{C}_x^{13}$ are finite for every $x \in \mathscr{C}_E$.*

1.7. **A “doubling” property.** Henceforth, we assume that $\mathscr{C}_x^{23}$ is infinite and $\mathscr{C}_x^{12}$, $\mathscr{C}_x^{13}$ are finite (for every $x \in \mathscr{C}_E$). The proof for the remaining cases is similar.

Lemma 1.11. *There exists $\rho_0 \in (0, 1)$ such that for every $x \in \mathscr{C}_E$, there exists $\rho_x \in I_0 := (-1, -\rho_0) \cup (\rho_0, 1)$ such that $\mathbf{u}_{23}(\rho_x) \in \mathscr{C}_x^{23}$.*

Proof. If not, using the continuity of $x \mapsto \mathscr{C}_x^{12}$, one can show that $\mathscr{C}_x^{12} = \{\mathrm{I}_3\}$ for some $x \in \mathscr{C}_E$. A contradiction. □

Fix such a ρ_0 and I_0. Using the A-action, one gets

Corollary 1.12. *For every $x \in \mathscr{C}_E$ and every $\lambda > 0$, there exists $\rho_x(\lambda) \in \lambda I_0$ such that $\mathbf{u}_{23}(\rho_x(\lambda)) \in \mathscr{C}_x^{23}$.*

Without loss of generality, assume that $\mathscr{C}_x^{23}$ is infinity for every $x \in \mathscr{C}_E$ and $I_0 = (\rho_0, 1)$.

1.8. **Unipotent blowup/Low entropy method.** The following calculation is the key to the low entropy method. Its use in dynamics can be traced back to the work of Ratner on joinings of unipotent flows.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -r \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} g_{11} & g_{12} & g_{13} - r g_{12} \\ g_{21} + r g_{31} & g_{22} + r g_{32} & g_{23} - r^2 g_{32} + r(g_{33} - g_{22}) \\ g_{31} & g_{32} & g_{33} - r g_{32} \end{bmatrix}$$

For simplicity, let $r_0 := \mathrm{InjRad}(\mathscr{C}_E) > 0$.

For a pair of points $x, x' \in \mathscr{C}_E$ with $d(x, x') < r_0$, there exists a unique $g = g(x, x') \in B(r_0)$ such that $x' = g.x$. We let

$$\varepsilon(x, x') := \|\mathrm{I}_3 - g(x, x')\|_{\sup}.$$

For $\delta > 0$, we let

$$r_\delta(x, x') := \min\left\{ \frac{\delta}{|g_{12}|}, \frac{\delta}{|g_{31}|}, \frac{\sqrt{\delta}}{\sqrt{|g_{32}|}}, \frac{\delta}{|g_{33} - g_{22}|} \right\}.$$

If some denominator is zero, we think of the corresponding term as being $+\infty$. So $r_\delta(x, x') \in (0, +\infty]$. From the above matrix calculation, we see that

Lemma 1.13. *For $x, x' \in \mathscr{C}_E$ with $d(x, x') < r_0$, If $g(x, x') \notin \mathrm{Z}_{\mathbf{SL}_3}(\mathrm{U}_{23})$, then $r_\delta(x, x') < +\infty$.*

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[7] make sense the these implications!

[8] Imagine two compact A-orbit are linked by a unipotent, then suitable $a_n \in \mathrm{A}$ would bring these two tori closer and closer, which is impossible

Assume $r_\delta(x,x') < +\infty$, we let $n = n_\delta(x,x')$ be the unique integer such that

$$r_\delta(x,x') \in [\rho_0^{-n}, \rho_0^{-(n+1)}).$$

$n_\delta(x,x')$ is large if $\varepsilon(x,x')$ is much smaller compared to δ.

By Corollary 1.12, we can find

$$r' = r'_\delta(x,x') \in [\rho_0^{-n}, \rho_0^{-(n+1)}) \text{ such that } \mathbf{u}_{23}(r') \in \mathscr{C}_x^{23},$$
$$r'' = r''_\delta(x,x') \in [\rho_0^{-(n+2)}, \rho_0^{-(n+3)}) \text{ such that } \mathbf{u}_{23}(r'') \in \mathscr{C}_x^{23}.$$

Also let

$$\lambda_\delta(x,x') := \frac{r''_\delta(x,x')}{r'_\delta(x,x')} \in (\rho_0^{-1}, \rho_0^{-3}).$$

For $s \in \mathbb{R}$, let

$$g(s) := \mathbf{u}_{23}(s) g \mathbf{u}_{23}(s)^{-1} = \begin{bmatrix} g(s)_{11} & g(s)_{12} & g(s)_{13} \\ g(s)_{21} & g(s)_{22} & g(s)_{23} \\ g(s)_{31} & g(s)_{32} & g(s)_{33} \end{bmatrix}$$
$$= \begin{bmatrix} g_{11} & g_{12} & g_{13} - sg_{12} \\ g_{21} + sg_{31} & g_{22} + sg_{32} & g_{23} - s^2 g_{32} + s(g_{33} - g_{22}) \\ g_{31} & g_{32} & g_{33} - sg_{32} \end{bmatrix}$$

Lemma 1.14. *Fix $\delta \in (0,1)$. Take $x, x' \in \mathscr{C}_E$ with $d(x,x') < r_0$ and $r_\delta(x,x') < +\infty$. Assume further that $\varepsilon(x,x') < \frac{\rho_0(1-\rho_0)}{4}\delta < \frac{1}{4}\rho_0\delta$. Then there is $s := s_\delta(x,x') \in \{r'_\delta(x,x'), r''_\delta(x,x')\}$ such that*

$$3\delta > \max\{|g(s)_{21}|, |g(s)_{13}|, |g(s)_{23}|\} \geq \rho_1\delta$$

where $\rho_1 := \frac{\rho_0(1-\rho_0)}{4}$.

Proof. The "$3\delta >$" part is easy. Let us focus on the other inequality.

If $r_\delta(x,x') = \delta |g_{12}|^{-1}$, then take $s := r'_\delta(x,x')$. We have

$$|g(s)_{13}| = |g_{13} - sg_{12}| \geq \rho_0\delta - \varepsilon(x,x') \geq \rho_1\delta.$$

Similarly, if $r = r_\delta(x,x') = \delta |g_{31}|^{-1}$, then

$$|g(s)_{21}| = |g_{21} + sg_{31}| \geq \rho_0\delta - \varepsilon(x,x') \geq \rho_1\delta.$$

where $s := r'_\delta(x,x')$.

Now assume that $r = r_\delta(x,x') = \min\left\{\delta |g_{32}|^{-\frac{1}{2}}, \delta |g_{33} - g_{22}|^{-1}\right\}$, then

$$\max\left\{(r')^2 |g_{32}|, r' |g_{33} - g_{22}|\right\} \geq \rho_0\delta.$$

where $r' := r'_\delta(x,x')$ Write $\lambda := \lambda_\delta(x,x')$ and note that

$$\begin{bmatrix} 1 & 1 \\ \lambda^2 & \lambda \end{bmatrix} \begin{bmatrix} -r^2 g_{32} \\ r(g_{33} - g_{22}) \end{bmatrix} = \begin{bmatrix} -r^2 g_{32} + r(g_{33} - g_{22}) \\ -(\lambda r)^2 g_{32} + (\lambda r)(g_{33} - g_{22}) \end{bmatrix}$$
$$\implies \begin{bmatrix} -r^2 g_{32} \\ r(g_{33} - g_{22}) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \lambda^2 & \lambda \end{bmatrix}^{-1} \begin{bmatrix} -r^2 g_{32} + r(g_{33} - g_{22}) \\ -(\lambda r)^2 g_{32} + (\lambda r)(g_{33} - g_{22}) \end{bmatrix}$$
$$\implies \left\| \begin{bmatrix} -r^2 g_{32} \\ r(g_{33} - g_{22}) \end{bmatrix} \right\|_{\sup} \leq 2 \left\| \begin{bmatrix} 1 & 1 \\ \lambda^2 & \lambda \end{bmatrix}^{-1} \right\|_{\sup} \left\| \begin{bmatrix} -r^2 g_{32} + r(g_{33} - g_{22}) \\ -(\lambda r)^2 g_{32} + (\lambda r)(g_{33} - g_{22}) \end{bmatrix} \right\|_{\sup}$$

But

$$\begin{bmatrix} 1 & 1 \\ \lambda^2 & \lambda \end{bmatrix}^{-1} = \frac{1}{\lambda - \lambda^2} \begin{bmatrix} \lambda & -1 \\ -\lambda^2 & 1 \end{bmatrix}^{-1}$$
$$\implies \left\| \begin{bmatrix} 1 & 1 \\ \lambda^2 & \lambda \end{bmatrix}^{-1} \right\|_{\sup} = \frac{\lambda^2}{\lambda^2 - \lambda} \leq \frac{1}{1 - \rho_0}.$$

So we have

$$\left\| \begin{bmatrix} -r^2 g_{32} + r(g_{33} - g_{22}) \\ -(\lambda r)^2 g_{32} + (\lambda r)(g_{33} - g_{22}) \end{bmatrix} \right\|_{\sup} \geq \frac{\rho_0(1-\rho_0)}{2}\delta.$$

Therefore,

$$\max\{|g(r)_{23}|, |g(\lambda r)_{23}|\} \geq \frac{\rho_0(1-\rho_0)}{2}\delta - \varepsilon(x,x') \geq \rho_1\delta.$$

□

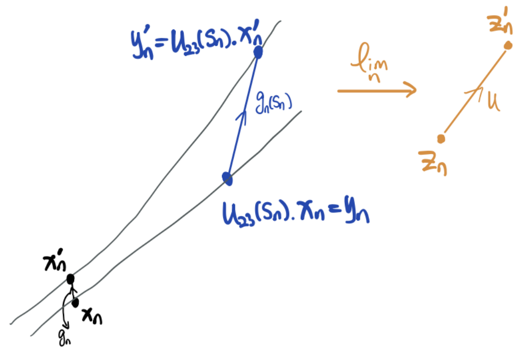
On the other hand, it is direct to verify that:

Lemma 1.15. *Assumption as in last lemma. Also, $s = s_\delta(x, x')$ same as there. Then*

$$\max\{|g(s)_{11} - 1|, |g(s)_{12}|, |g(s)_{31}|, |g(s)_{32}|\} \le \varepsilon(x, x');$$
$$\max\{|g(s)_{22} - 1|, |g(s)_{33} - 1|\} \le 2\sqrt{\varepsilon(x, x')}.$$

If one imagines that when $d(x, x')$ (and hence $\varepsilon(x, x')$) is extremely small (compared to δ, ρ_0,...), the matrix $g(s)$ would look like a unipotent matrix in the centralizer of U_{23}:

$$\begin{bmatrix} \approx 1 & \approx 0 & g(s)_{13} \\ g(s)_{21} & \approx 1 & g(s)_{23} \\ \approx 0 & \approx 0 & \approx 1 \end{bmatrix}$$



1.9. **Take the limit.** For $t \in \mathbb{R}$, let

$$\beta_t := \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix},$$

which commutes with U_{23}.

We first find, by pigeon-hole principle, a sequence of pairs $(x_n, x'_n) \in \mathscr{C}_E \times \mathscr{C}_E$ with

$$x'_n = \beta_t.x_n \text{ for some } t \in \mathbb{R} \text{ with } t \ge 100;$$
$$d(x_n, x'_n) < \mathrm{InjRad}(\mathscr{C}_E) \text{ converges to } 0. \tag{3}$$

Fix some point $x_0 \in \mathscr{C}_E$. For $n \in \mathbb{Z}^+$, cover $\mathscr{C}_E$ by finitely many balls of diameter $< \min\{n^{-1}, \mathrm{InjRad}(\mathscr{C}_E)\}$, then we can find $m < m' \in \mathbb{Z}$ such that $x_n := \beta_{100m}.x_0$ and $x'_n := \beta_{100m'}.x_0$ lies in the same ball. This pair of (x_n, x'_n) satisfies Equa.(3) above.

Thus, there exists a unique $g_n := g(x_n, x'_n) \in \mathbf{SL}_3(\mathbb{R})$ with $x'_n = g_n.x_n$ and $d(x_n, x'_n) = d(\mathrm{I}_3, g_n)$. Further assume that

$$g_n \notin \mathrm{Z}_{\mathbf{SL}_3}(\mathrm{U}_{23}) \tag{4}$$

Consequently, $r_\delta(x_n, x'_n) \neq +\infty$. By Lemma 1.17 below, this assumption is satisfied as long as $d(x_n, x'_n)$ is small enough.

Now choose $\delta > 0$. For n large enough (such that $\varepsilon(x_n, x'_n) < 0.5(1 - \rho_0)\delta$ and $g(x_n, x'_n) \notin \mathrm{Z}_{\mathbf{SL}_3(\mathbb{R})}(\mathrm{U}_{23})$), apply the unipotent blowup as in the last subsection.

Define

$$y_{n,\delta} := \mathbf{u}_{23}(s_{n,\delta}).x_n,\ y'_{n,\delta} := \mathbf{u}_{23}(s_{n,\delta}).x'_n.$$

Then

$$y'_{n,\delta} = g_{s_{n,\delta}}.y_{n,\delta}$$

with (write $s = s_{n,\delta}$ and $\varepsilon := \varepsilon(x_n, x'_n)$ for simplicity)

$$g(s_{n,\delta}) - \mathrm{I}_3 = \begin{bmatrix} \le \varepsilon & \le \varepsilon & g(s)_{13} \\ g(s)_{21} & \le 2\sqrt{\varepsilon} & g(s)_{23} \\ \le \varepsilon & \le \varepsilon & \le 2\sqrt{\varepsilon} \end{bmatrix}$$

with

$$\max\{|g(s_{n,\delta})_{13}|, |g(s_{n,\delta})_{21}|, |g(s_{n,\delta})_{23}|\} \ge \rho_1\delta.$$

Passing to a subsequence, assume

$$\lim y_{n,\delta} = z_\delta,\ \lim y'_{n,\delta} = z'_\delta,\ \lim g(s_{n,\delta}) = u(\delta)$$

exists. Then $z_\delta, z'_\delta \in \mathscr{C}_E$ and $z'_\delta = u(\delta).z_\delta$ where

$$u(\delta) = \begin{bmatrix} 1 & 0 & u(\delta)_{13} \\ u(\delta)_{21} & 1 & u(\delta)_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

with

$$\rho_1 \delta \leq \max\{|u(\delta)_{13}|, |u(\delta)_{21}|, |u(\delta)_{23}|\} \leq 3\delta.$$

By Assumption 1.2, Corollary 1.7 and Lemma 1.9, $u(\delta)_{21} = u(\delta)_{13} = 0$.

By the continuity of $\mathscr{C}^{23}_{\bullet}$, we have (note that $x'_n = \beta_t.x_n$ and β_t centralizes U_{23})

$$\mathscr{C}^{23}_{x_n} = \mathscr{C}^{23}_{x'_n} \implies \mathscr{C}^{23}_{y_{n,\delta}} = \mathscr{C}^{23}_{y'_{n,\delta}} \implies \mathscr{C}^{23}_{z_\delta} = \mathscr{C}^{23}_{z'_\delta}.$$

But $u(\delta) \in \mathrm{U}_{23}$, hence

$$u(\delta) \cdot \mathscr{C}^{23}_{z'_\delta} = \mathscr{C}^{23}_{z_\delta}.$$

Thus $\mathscr{C}^{23}_{z_\delta}$ is invariant under translation by $u(\delta)$. By taking a limit point z of (z_δ) as $\delta \to 0$, we see that, by continuity, $\mathrm{U}_{23}.z \subset \mathscr{C}_E$. Hence $(\mathrm{A}\mathrm{U}_{23}).z$ is bounded. But this is impossible.

1.10. **No exceptional returns.** We verify Equa.(4) from last subsection. To have slightly better-looking notation, we replace the index $(2,3)$ by $(1,3)$ and β_t is replaced by $\beta'_t := \mathrm{diag}(e^{-t}, e^{2t}, e^{-t})$ accordingly.

Lemma 1.16. *If $M \in \mathbf{SL}_3(\mathbb{Z})$ only has two different eigenvalues, then all eigenvalues of M are ± 1.*

Proof. Let $p(x) := \det(x\mathrm{I}_3 - M) \in \mathbb{Z}[x]$ be the characteristic polynomial of M. By assumption, at least two roots of $p(x)$ are the same. Then $p(x)$ is reducible in $\mathbb{Q}[x]$. If you have not learned Galois theory, then here is a direct way of seeing this. Write $p(x) = (x-\alpha)^2(x-\beta) = x^3 + Ax^2 + Bx + C$ for some $\alpha, \beta \in \mathbb{R}$, $A, B, C \in \mathbb{Q}$. By comparing the coefficients, we see that

$$A = -x_2 - 2x_1,\ B = x_1^2 + 2x_1x_2.$$

The first one implies that $2Ax_1 = -2x_1x_2 - 4x_1^2$, combined with the second one, we get

$$x_1^2 + 2/3Ax_1 + 1/3B = 0.$$

By Euclidean algorithm, the polynomial $q(x) := x^2 + 2/3Ax + 1/3B$ divides $p(x)$. In particular, $p(x)$ is reducible in $\mathbb{Q}[x]$.

Note that $p(x)$ is also reducible in $\mathbb{Z}[x]$ by Gauss lemma. Write $p(x) = (x^2+ax+b)(x-c)$ for some $a, b, c \in \mathbb{Z}$. Since $\det M = 1$, we have $bc = 1$. So $b = c = 1$ or $b = c = -1$. If $x^2 + ax + b$ is irreducible, then it would have two different non-rational roots. So all three roots of p are distinct, contradiction. Hence $p(x) = (x-x_1)(x-x_2)(x-x_3)$ for some $x_i \in \mathbb{Z}$ with $\prod x_i = 1$. So all $x_i = \pm 1$. ☐

Lemma 1.17. *Take $x \in \mathrm{X}_3$ be such that $\mathrm{A}.x$ is bounded. Assume $\eta \in (0, \mathrm{InjRad}(x))$ is small enough such that*

$$d(\mathrm{I}_3, g) < \eta \implies \|\mathrm{I}_3 - g\|_{\sup} < 0.1.$$

Let $t \geq 100$ be such that $\beta'_t.x = g.x$ with $d(x, g.x) = d(\mathrm{I}_3, g) < \eta$. Then g is not contained in the centralizer of U_{13}. Namely, it is impossible for g to take the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} e^{-s} & 0 & 0 \\ 0 & e^{2s} & 0 \\ 0 & 0 & e^{-s} \end{bmatrix} \cdot \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} e^{-s} & 0 & 0 \\ 0 & e^{2s} & 0 \\ 0 & 0 & e^{-s} \end{bmatrix} \cdot \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

Proof. Assume g does take this form and let us derive a contradiction.

By assumption on η, $\mathrm{diag}(1, -1, -1)$ is not allowed. So we have

$$\begin{bmatrix} e^{-s} & 0 & 0 \\ 0 & e^{2s} & 0 \\ 0 & 0 & e^{-s} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} .x = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} .x$$

Thus,

$$\begin{bmatrix} e^{-(s-t)} & 0 & 0 \\ 0 & e^{2(s-t)} & 0 \\ 0 & 0 & e^{-(s-t)} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \text{ is conjugate to some element in } \mathbf{SL}_3(\mathbb{Z}).$$

By Lemma 1.16, $s = t$. But our assumption implies that $t \geq 100 > \log(1.1) \geq s$. Contradiction. □

References

[EKL06] Manfred Einsiedler, Anatole Katok, and Elon Lindenstrauss, *Invariant measures and the set of exceptions to Littlewood's conjecture*, Ann. of Math. (2) **164** (2006), no. 2, 513–560. MR 2247967 1