LECTURE 4

RUNLIN ZHANG†

Contents

Notation

1. Lecture 4, Space of lattices in $\mathbb{R}^3$, rigidity conjectures and an Isolation principle

1.1. **Prelude.** The connection between Littlewood conjecture (and Oppenheim conjecture) and subgroup action on the space of lattices has been (at least implicitly) noted in Cassels–Swinnerton-Dyer's paper [CSD55] in 1950s. Whereas Oppenheim conjecture is now a theorem of Margulis, Littlewood conjecture remains unsolved. One contribution of the CSD paper is an "isolation principle". This can be used to establish implications between several (unknown) conjectures. Also, it can be used to show that Littlewood conjecture holds for pairs of numbers contained in the same cubic number field. One may also consult [Mar97] and [LW01].

1.2. **Space of unimodular lattices.**

1.2.1. *The definition.* Just as before, we say that a discrete subgroup Λ of $(\mathbb{R}^3, +)$ is a lattice iff $\Lambda = \mathbb{Z}\mathbf{u} + \mathbb{Z}\mathbf{v} + \mathbf{w}$ for some linearly independent $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. For such a lattice $\Lambda = \mathbb{Z}\mathbf{u} + \mathbb{Z}\mathbf{v} + \mathbb{Z}\mathbf{w}$, define $\|\Lambda\| := |\det((\mathbf{u}, \mathbf{v}, \mathbf{w}))|$. A lattice Λ is said to be unimodular iff $\|\Lambda\| = 1$.

Let X_3 denote the set of all unimodular lattices of $\mathbb{R}^3$ equipped with the Chabauty topology. Under this topology, a sequence of unimodular lattices (Λ_n) converges to $\Lambda \in \mathrm{X}_2$ iff we can write $\Lambda_n = \mathbb{Z}\mathbf{u}_n + \mathbb{Z}\mathbf{v}_n + \mathbb{Z}\mathbf{w}_n$ and $\Lambda = \mathbb{Z}\mathbf{u} + \mathbb{Z}\mathbf{v} + \mathbb{Z}\mathbf{w}$ such that $\|\mathbf{u}_n - \mathbf{u}\|, \|\mathbf{v}_n - \mathbf{v}\|, \|\mathbf{w}_n - \mathbf{w}\| \to 0$.

1.2.2. *Mahler's criterion.* Define

$$\mathrm{sys}(\Lambda) := \inf_{\mathbf{v} \neq \mathbf{0} \in \Lambda} \|\mathbf{v}\|.$$

Theorem 1.1. *The function* $\mathrm{sys} : \mathrm{X}_3 \to (0, +\infty)$ *is bounded, continuous and proper.*

† Email: zhangrunlinmath@outlook.com.

1.2.3. *Group action and local coordinates.* The group $\mathbf{SL}_3(\mathbb{R})$, consisting of 3-by-3 real matrices of determinant one, acts on X_3 naturally. This action is continuous and transitive. The map $g \mapsto \mathbb{Z}^3$ induces a homeomorphism $\mathbf{SL}_3(\mathbb{R})/\mathbf{SL}_3(\mathbb{Z}) \cong \mathrm{X}_3$.

We define several subgroups of $\mathbf{SL}_3(\mathbb{R})$:

$$\mathrm{A} := \left\{ \begin{bmatrix} e^{t_1} & 0 & 0 \\ 0 & e^{t_2} & 0 \\ 0 & 0 & e^{t_3} \end{bmatrix} \middle| \sum t_i = 0,\ t_i \in \mathbb{R} \right\};$$

$$\mathrm{U}^{++} := \left\{ \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \middle| u_{ij} \in \mathbb{R} \right\}; \quad \mathrm{U}^{--} := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ u_{21} & 1 & 0 \\ u_{31} & u_{32} & 1 \end{bmatrix} \middle| u_{ij} \in \mathbb{R} \right\}$$

For $s \in \mathbb{R}$ and $i \neq j = 1,2,3$, let $\mathbf{u}_{ij}(s) - \mathrm{I}_3$ be the matrix whose (i,j)-th entry is equal to s and is zero otherwise. Note that $\mathbf{u}_{ij}(\mathbb{R})$ is a subgroup of $\mathbf{SL}_3(\mathbb{R})$ isomorphic to $(\mathbb{R},+)$.

For $\varepsilon > 0$, let

$$\mathrm{A}(\varepsilon) := \left\{ \begin{bmatrix} e^{t_1} & 0 & 0 \\ 0 & e^{t_2} & 0 \\ 0 & 0 & e^{t_3} \end{bmatrix} \middle| \sum t_i = 0,\ t_1, t_2 \in (-\varepsilon, \varepsilon) \right\};$$

$$\mathrm{U}^{++}(\varepsilon) := \left\{ \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \middle| u_{ij} \in (-\varepsilon, \varepsilon) \right\};$$

$$\mathrm{U}^{--}(\varepsilon) := \left\{ \begin{bmatrix} 1 & 0 & 0 \\ u_{21} & 1 & 0 \\ u_{31} & u_{32} & 1 \end{bmatrix} \middle| u_{ij} \in (-\varepsilon, \varepsilon) \right\}.$$

1.2.4. *Local coordinates.* For $x \in \mathrm{X}_3$, let $\mathrm{Obt}_x : \mathbf{SL}_3(\mathbb{R}) \to \mathrm{X}_3$ be the orbit map $g \mapsto g.x$.

For every compact subset $\mathscr{C} \subset \mathrm{X}_3$, there exists $\varepsilon > 0$ such that for every $x \in \mathscr{C}$,

$$\begin{aligned} \mathrm{A}(\varepsilon) \times \mathrm{U}^{--}(\varepsilon) \times \mathrm{U}^{++}(\varepsilon) &\to \mathrm{X}_3 \\ (a, v, u) &\mapsto \mathrm{Obt}(a \cdot v \cdot u).x \end{aligned} \tag{1}$$

is a homeomorphism onto an open neighborhood, termed $\mathcal{N}_x^{AU}(\varepsilon)$, of $x \in \mathrm{X}_3$. Likewise, for $\varepsilon > 0$ small enough, we define $\mathcal{N}_x^{UA}(\varepsilon)$ by using $\mathrm{Obt}_x(v \cdot u \cdot a) = vua.x$ for $u \in \mathrm{U}^{--}$, $v \in \mathrm{U}^{++}$ and $a \in \mathrm{A}$.

1.2.5. *A metric.* One can define a right-invariant metric on $\mathbf{SL}_3(\mathbb{R})$ by

$$\mathrm{dist}(g,h) := \log\left(1 + \left\|gh^{-1}\right\|_{\mathrm{op}} + \left\|hg^{-1}\right\|_{\mathrm{op}}\right).$$

It induces a metric on $\mathbf{SL}_3(\mathbb{R})/\mathbf{SL}_3(\mathbb{Z}) \cong \mathrm{X}_3$ by

$$\mathrm{dist}(g\mathbb{Z}^3, h\mathbb{Z}^3) := \inf_{\gamma \in \mathbf{SL}_3(\mathbb{Z})} \mathrm{dist}(g\gamma, h\gamma).$$

This metric is compatible with the topology given. For ε small enough depending on some compact set $\mathscr{C}$, the orbit map $g \mapsto g.x$ is an isometry (and in particular, a homeomorphism) from $B(\varepsilon) := \{g,\ d(g, \mathrm{id}) < \varepsilon\}$ to its image for every $x \in \mathscr{C}$.

1.2.6. *The invariant measure.* The group $\mathbf{SL}_3(\mathbb{R})$ has a bi-invariant locally finite measure $\mathrm{m}_{\mathbf{SL}_3(\mathbb{R})}$. After being normalized by a positive scalar, it induces an $\mathbf{SL}_3(\mathbb{R})$-invariant probability measure $\mathrm{m}_{\mathrm{X}_3}$ on X_3. For $\varepsilon > 0$ small enough, the orbit map $g \mapsto g.x$ identifies the measure $\mathrm{m}_{\mathbf{SL}_3(\mathbb{R})}$ restricted to $B(\varepsilon)$ with $\mathrm{m}_{\mathrm{X}_3}$ restricted to $B(\varepsilon).x$.

1.3. **Two problems in Diophantine approximations.** For a function $f : \mathbb{R}^3 \to \mathbb{R}$, let $m^*(f) := \inf\{f(x),\ x \in \mathbb{Z}^3,\ x \neq 0\}$.

First we consider real quadratic forms in three variables. Let $Q : \mathbb{R}^3 \to \mathbb{R}$ be such a form. So there are real numbers $(q_{ij})_{i,j=1,2,3}$ with $q_{ij} = q_{ji}$ such that $Q(x_1, ..., x_n) = \sum q_{ij} x_i x_j$.

Theorem 1.2. *Assume Q is non-degenerate (that is, $\det(q_{ij}) \neq 0$). If Q is indefinite and is not a scalar multiple of one with $\mathbb{Q}$-coefficients, then $m^*(f) = 0$.*

Remark 1.3. *Not true if 3 replaced by 2. True for forms of variables more than there, which can be reduced to the above case.*

Let $\phi : \mathbb{R}^3 \to \mathbb{R}$ be a product of three real linear forms. Namely, there exist three L_1, L_2, L_3 linear functionals on $\mathbb{R}^3$ such that $\phi(x) = L_1(x)L_2(x)L_3(x)$.

Conjecture 1.4. *Assume that ϕ is non-degenerate (namely, L_1, L_2, L_3 are linearly independent) and is not a scalar multiple of one with $\mathbb{Q}$-coefficients. Then $m^*(\phi) = 0$.*

1.4. **Linear symmetry.** Let Q, ϕ be as in the last section. Let

$$H_Q := \mathbf{SO}_Q(\mathbb{R}) := \left\{ g \in \mathbf{SL}_3(\mathbb{R}) \,\middle|\, Q(g.x) = Q(x),\ \forall x \in \mathbb{R}^3 \right\}.$$
$$H_\phi := \left\{ g \in \mathbf{SL}_3(\mathbb{R}) \,\middle|\, \phi(g.x) = \phi(x),\ \forall x \in \mathbb{R}^3 \right\}.$$

Lemma 1.5. *We have*

$$H_Q.\mathbb{Z}^3 \text{ is unbounded in } \mathrm{X}_3 \implies m^*(Q) = 0,$$
$$H_\phi.\mathbb{Z}^3 \text{ is unbounded in } \mathrm{X}_3 \implies m^*(\phi) = 0.$$

Lemma 1.6. *We have*

$$H_Q.\mathbb{Z}^3 \text{ is compact in } \mathrm{X}_3 \implies \text{up to a scalar, } Q \text{ has rational coefficients},$$
$$H_\phi.\mathbb{Z}^3 \text{ is compact in } \mathrm{X}_3 \implies \text{up to a scalar, } \phi \text{ has rational coefficients}.$$

Theorem 1.7. *Assume Q is indefinite. Every bounded orbit of H_Q on X_3 is closed (and hence compact).*

Conjecture 1.8. *Every bounded orbit of A on X_3 is closed (and hence compact).*

By the lemmas above, we have

Corollary 1.9. *Conjecture 1.8 $\implies$ Conjecture 1.4. And Theorem 1.7 $\implies$ Theorem 1.2.*

1.5. **Measure rigidity.** How to prove Theorem 1.7? A crucial fact is that the symmetry group $\mathbf{SO}_Q(\mathbb{R})$, locally isomorphic to $\mathbf{SL}_2(\mathbb{R})$, is generated by unipotent matrices. Though the original proof of Theorem 1.7 does not involve any measures, it is possible to decompose the proof of Theorem 1.7 into two steps:

1. Classification of unipotent-invariant ergodic measures: they are all homogeneous;
2. Deduce Theorem 1.7 from this.

Regarding A-action, the measure classification is unknown:

Conjecture 1.10. *Every* A*-invariant probability measure is a convex combination of those supported on compact* A*-orbits and* $\mathrm{m}_{\mathrm{X}_3}$*.*

Conjecture 1.11. *Every* A*-invariant compact subset of* X_3 *is a union of finitely many compact* A*-orbits.*

Conjecture 1.12. *Every bounded subset of* X_3 *contains only finitely many compact* A*-orbits.*

We do know the following implications

Theorem 1.13. *Conjecture 1.10 $\implies$ Conjecture 1.8 $\implies$ Conjecture 1.11 $\implies$ Conjecture 1.12.*

Also,

Theorem 1.14. *Conjecture 1.11 $\implies$ Littlewood conjecture.*

The proof of these implications is based on the following "isolation principle".

Theorem 1.15. *Given a compact* A*-orbit* $\mathrm{A}.y$*. For every compact subset $\mathscr{C} \subset \mathrm{X}_3$, there exists $\varepsilon > 0$ such that*

$$\mathrm{dist}(x, y) < \varepsilon \implies \mathrm{A}.x \not\subseteq \mathscr{C}$$

In particular, if the orbit closure of some A*-orbit* $\mathrm{A}.x$ *contains a compact* A*-orbit, then* $\mathrm{A}.x$ *is either compact or unbounded.*

Remark 1.16. *This (and all the conjectures above) is wrong on* X_2 *where* A*, isomorphic to* $(\mathbb{R}, +)$*, has "rank one".*

Conjecture 1.10 seems to be partly motivated by a question of Furstenberg [Fur67]. Let $T_p : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ defined by $x + \mathbb{Z} \mapsto px + \mathbb{Z}$. Note that there are many irrational numbers such that $\{T_p^n \alpha,\ n \in \mathbb{Z}^+\}$ is not dense in $\mathbb{R}/\mathbb{Z}$.

Theorem 1.17. *If α is irrational, then*

$$\left\{T_2^n T_3^m.\alpha \;\middle|\; n, m \in \mathbb{Z}^+\right\}$$

is dense in $\mathbb{R}/\mathbb{Z}$.

Conjecture 1.18. *Let μ be a probability measure on $\mathbb{R}/\mathbb{Z}$ invariant under T_2 and T_3, then μ is a convex combinations of those supported on certain finite sets and the Lebesgue measure.*

What we know about Conjecture 1.10 is

Theorem 1.19. *Let μ be an A-invariant probability measure with compact support, then $h_\mu(a) = 0$ for every $a \in \mathrm{A}$.*

This may be compared with (see [Rud90])

Theorem 1.20. *Let μ be an ergodic probability measure on $\mathbb{R}/\mathbb{Z}$ invariant under T_2 and T_3 and $h_\mu(T_2) > 0$, then μ is the Lebesgue measure.*

Applications of measure rigidity theorems can be found in the survey [Ein10] or [Lin22].

1.6. **Compact A-orbits.** In this section we give a more explicit description of compact A-orbits.

Lemma 1.21. *Let $\mathrm{A}g\mathbb{Z}^3$ be a compact A orbit. Then there exists a cubic number field (i.e. field extension of $\mathbb{Q}$ of degree three) K, $(x, y, z) \in K^3$ and $\lambda \in \mathbb{R}$ such that $\mathrm{A}g\mathbb{Z}^3 = \mathrm{A}M\mathbb{Z}^3$ for*

$$M = \lambda \cdot \begin{bmatrix} x & y & z \\ \sigma_2(x) & \sigma_2(y) & \sigma_2(z) \\ \sigma_3(x) & \sigma_3(y) & \sigma_3(z) \end{bmatrix} \in \mathbf{SL}_3(\mathbb{R}) \tag{2}$$

where $\{\mathrm{id}, \sigma_2, \sigma_3\}$ denotes the three field embeddings of K into $\mathbb{C}$.

Lemma 1.22. *Assume $\gamma \in \mathbf{SL}_3(\mathbb{Z})$ is diagonalizable and none of the eigenvalues are equal to ± 1. Then its characteristic polynomial is irreducible in $\mathbb{Q}[x]$.*

Proof. Let $p(x) \in \mathbb{Z}[X] := \det(x\mathrm{I}_3 - \gamma)$ be the characteristic polynomial of γ. It suffices to show that $p(x)$ is irreducible in $\mathbb{Z}[x]$ as it is monic (Gauss' lemma?). Otherwise,

$$p(x) = (x^2 + ax + b)(x + c), \quad \exists\, a, b, c \in \mathbb{Z}$$

Since $\det(\gamma) = 1$, $bc = 1$. So $c = \pm 1$, a contradiction. $\square$

Proof of Lemma 1.21. By assumption, $\mathrm{A}g\,\mathbf{SL}_3(\mathbb{Z})/\,\mathbf{SL}_3(\mathbb{Z})$ is compact. In other words, $\mathrm{A} \cap g\,\mathbf{SL}_3(\mathbb{Z})g^{-1}$ is a lattice in A. Therefore, we can find $\gamma \in g^{-1}\mathrm{A}g \cap \mathbf{SL}_3(\mathbb{Z})$ with three distinct eigenvalues and none of which is equal to ± 1. Let $p(x)$ be the characteristic polynomial of γ, then $p(x)$ is irreducible by lemma above. Let θ be one of its root. Then $K := \mathbb{Q}(x)$, isomorphic to $\mathbb{Q}[x]/(p(x))$, has dimension three as a $\mathbb{Q}$-vector space. So there exists exactly three different embeddings $\{\mathrm{id}, \sigma_2, \sigma_3\}$ of K into $\mathbb{C}$. By linear algebra, one can find $(x, y, z) \in K^3$ with

$$(x, y, z) \cdot \gamma = \theta(x, y, z)$$

By applying the other two embeddings, we get

$$\begin{bmatrix} x & y & z \\ \sigma_2(x) & \sigma_2(y) & \sigma_2(z) \\ \sigma_3(x) & \sigma_3(y) & \sigma_3(z) \end{bmatrix} \cdot \gamma = \begin{bmatrix} \theta & 0 & 0 \\ 0 & \sigma_2(\theta) & 0 \\ 0 & 0 & \sigma_3(\theta) \end{bmatrix} \cdot \begin{bmatrix} x & y & z \\ \sigma_2(x) & \sigma_2(y) & \sigma_2(z) \\ \sigma_3(x) & \sigma_3(y) & \sigma_3(z) \end{bmatrix}$$

Define M as in Equa.(2) where λ is chosen such that M has determinant one. Then $M\gamma M^{-1}$, as well as $g\gamma g^{-1}$, belongs to A. Replacing θ by $\sigma_i(\theta)$ and K by $\sigma_i(K)$ if necessary, we assume that

$$M\gamma M^{-1} = g\gamma g^{-1}.$$

Consequently, gM^{-1} commutes with $M\gamma M^{-1}$ and is therefore diagonal. In particular, $\mathrm{A}g\mathbb{Z}^3 = \mathrm{A}M.\mathbb{Z}^3$. This finishes the proof. $\square$

1.7. **An equivalent form of Littlewood conjecture.** Let

$$\mathrm{A}^+ := \left\{ \begin{bmatrix} e^{t_1} & 0 & 0 \\ 0 & e^{t_2} & 0 \\ 0 & 0 & e^{t_3} \end{bmatrix} \middle| \sum t_i = 0,\ t_1, t_2 > 0 \right\}$$

be a sub-semigroup of A.

For a pair of real numbers $(\alpha, \beta) \in \mathbb{R}^2$, let

$$\Lambda_{\alpha,\beta} := \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} .\mathbb{Z}^3 = \mathbb{Z} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \mathbb{Z} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \mathbb{Z} \begin{bmatrix} \alpha \\ \beta \\ 1 \end{bmatrix}.$$

Lemma 1.23. *Let $(\alpha, \beta) \in \mathbb{R}^2$. The following two are equivalent*

(1) $\mathrm{A}^+.\Lambda_{\alpha,\beta}$ *is unbounded in* X_3*;*

(2) (α, β) *satisfies Littlewood conjecture.*

Proof. From definition we have

$$\begin{bmatrix} e^{t_1} & 0 & 0 \\ 0 & e^{t_2} & 0 \\ 0 & 0 & e^{t_3} \end{bmatrix} \Lambda_{\alpha,\beta} = \left\{ \begin{bmatrix} e^{t_1}(l + n\alpha) \\ e^{t_2}(m + n\beta) \\ e^{-t_1-t_2} n \end{bmatrix} \middle| l, m, n \in \mathbb{Z} \right\}$$

Take $\varepsilon \in (0, 1)$.

If $(t_1, t_2) \in (\mathbb{R}^+)^2$ is such that $\mathrm{sys}(\mathbf{a}_{t_1,t_2}\Lambda_{\alpha,\beta}) < \varepsilon$, then we can find $(l, m, n) \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}$ such that

$$\left. \begin{array}{r} |e^{t_1}(l + n\alpha)| < \varepsilon \\ |e^{t_2}(m + n\beta)| < \varepsilon \\ |e^{-t_1-t_2} n| < \varepsilon \end{array} \right\} \implies \left\{ \begin{array}{l} |n|\,|l + n\alpha|\,|m + n\beta| < \varepsilon^3, \\ n \neq 0 \end{array} \right.$$

Hence $n \neq 0$ and $|n|\,\langle n\alpha \rangle \langle n\beta \rangle < \varepsilon^3$.

Conversely, let $n \in \mathbb{Z}_{\neq 0}$ be such that $|n|\,\langle n\alpha \rangle \langle n\beta \rangle < \varepsilon^3$. Then one finds l, m such that $\langle n\alpha \rangle = |l + n\alpha|$ and $\langle n\beta \rangle = |m + n\beta|$. Assume $l + n\alpha \neq 0$ and $m + n\beta \neq 0$ (the remaining cases are left to the reader). We wish to set $t_1, t_2 \in \mathbb{R}$ such that

$$e^{t_1} = \frac{\varepsilon}{|l + n\alpha|},\ e^{t_2} = \frac{\varepsilon}{|m + n\alpha|}. \tag{3}$$

But there is no guarantee that $t_1, t_2 > 0$, which happens exactly when one of $\langle n\alpha \rangle$ or $\langle n\alpha \rangle$ is larger than ε. This can be remedied as follows:

Say $\langle n\beta \rangle > \varepsilon$. By Dirichlet theorem, we can find $n_2 < \lceil \varepsilon^{-1} \rceil$ such that

$$\langle n_2 n\beta \rangle < \lceil \varepsilon^{-1} \rceil^{-1} < \varepsilon.$$

On the other hand,

$$|n|\,\langle n_2 n\alpha \rangle \leq |nn_2|\,\langle n\alpha \rangle < (\varepsilon^{-1} + 1)\varepsilon^2 < 2\varepsilon.$$

Thus, if replacing n by $n' := nn_2$ and ε by $\varepsilon' := \sqrt[3]{2\varepsilon^2}$, we would have t_1, t_2 as defined by Equa.(3) are both positive. One has

$$\left| e^{t_1}(l + n\alpha) \right| = \varepsilon',\ \left| e^{t_2}(m + n\beta) \right| = \varepsilon',\ \left| e^{-t_1-t_2} n \right| < \varepsilon'.$$

And the proof is complete. □

1.8. **Conjecture 1.11 implies Littlewood.** By Lemma 1.23, it suffices to show that $\mathrm{A}^+.\Lambda_{\alpha,\beta}$ is not bounded. So let us assume that it is and seek for a contradiction.

Define

$$Y := \left\{ y \in \mathrm{X}_3 \;\middle|\; y = \lim \mathbf{a}_{(s_n,t_n)}.\Lambda_{\alpha,\beta},\ \exists\, s_n, t_n \to +\infty \right\}.$$

Then Y is A-invariant and bounded. Let $\overline{Y}$ be its closure, which is also A-invariant. By Conjecture 1.23, $\overline{Y}$ is a finite union of compact A-orbits. Therefore, Y is also a finite union of compact A-orbits, say

$$Y = \mathrm{A}y_1 \sqcup \mathrm{A}y_2 \sqcup \ldots \sqcup \mathrm{A}y_k.$$

Choose $\varepsilon > 0$ small enough such that $\mathcal{N}_{\mathrm{A}y_i}(\varepsilon)$ for $i = 1, \ldots, k$ are disjoint from each other. On the other hand, by the definition of Y, there exists $T(\varepsilon) \in \mathbb{R}^+$ such that

$$Y_N := \left\{ \mathbf{a}_{(s,t)} \;\middle|\; s, t > T(\varepsilon) \right\} \subset \bigsqcup_{i=1}^{k} \mathcal{N}_{\mathrm{A}y_i}(\varepsilon).$$

But Y_N is connected, it has to be contained in a unique $\mathcal{N}_{\mathrm{A}y_i}(\varepsilon)$. In other words, $k=1$ and $Y=\mathrm{A}.y_1$.

Using local coordinates, one shows that

Lemma 1.24. *For $\varepsilon>0$ small enough, the map*

$$\begin{aligned} \mathrm{U}^{--}(\varepsilon)\times\mathrm{U}^{++}(\varepsilon)\times\mathrm{A}.y_1 &\to \mathrm{X}_3 \\ (v,u,a.y_1) &\mapsto vua.y_1 \end{aligned}$$

is a homeomorphism onto an open subset, called $\mathcal{N}^{UA}_{\mathrm{A}.y_1}(\varepsilon)$.

Choose $\varepsilon>0$ small enough according to this lemma and find N large enough such that $Y_N\subset\mathcal{N}^{UA}_{\mathrm{A}.y_1}(0.5\varepsilon)$. Note that Y_N is A^+-invariant, so we can analyze Y_N under the action of A^+ using these local coordinates. For $z=\mathbf{u}^{--}(z)\mathbf{u}^{++}(z)y_z\in Y_N$ for some $y_z\in\mathrm{A}.y_1$ and $a\in\mathrm{A}^+$,

$$a.z=(a\mathbf{u}^{--}(z)a^{-1})\cdot(a\mathbf{u}^{++}(z)a^{-1})\cdot a.y_z.$$

If $\mathbf{u}^{++}(z)\neq\mathrm{I}_3$, then one can find $a\in\mathrm{A}^+$ such that $a\mathbf{u}^{++}(z)a^{-1}\in\mathrm{U}_\varepsilon\setminus\mathrm{U}_{0.5\varepsilon}$. This is a contradiction. Likewise, we also have that the $(2,1)$-entry of $\mathbf{u}^{--}(z)$ is zero. Combined with Lemma 1.21, we get

$$\begin{bmatrix}1&\alpha&\beta\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}e^{t_1}&0&0\\0&e^{t_2}&0\\0&0&e^{t_3}\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\r_1&r_2&1\end{bmatrix}\begin{bmatrix}x&y&z\\\sigma_2(x)&\sigma_2(y)&\sigma_2(z)\\\sigma_3(x)&\sigma_3(y)&\sigma_3(z)\end{bmatrix}\cdot\gamma$$

for some

$$\gamma\in\mathbf{SL}_3(\mathbb{Z}),\ t_1,t_2,t_3,r_1,r_2\in\mathbb{R},\ x,y,z\in\text{some cubic number field }K.$$

Hence

$$\begin{bmatrix}1&\alpha&\beta\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}e^{t_1}&0&0\\0&e^{t_2}&0\\0&0&e^{t_3}\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\r_1&r_2&1\end{bmatrix}\begin{bmatrix}x'&y'&z'\\\sigma_2(x')&\sigma_2(y')&\sigma_2(z')\\\sigma_3(x')&\sigma_3(y')&\sigma_3(z')\end{bmatrix}$$

for some possibly different $x',y',z'\in K$. By comparing the second row of both sides, one sees that $x'=z'=0$, which is a contradiction.

1.9. **Conjecture 1.8 implies Conjecture 1.11.** Assume otherwise, then we can find infinitely many distinct compact A-orbits $\mathrm{A}.y_1,\mathrm{A}y_2,...$ contained in some fixed compact subset $\mathscr{C}\subset\mathrm{X}_3$. Let y be a limit point of $(\mathrm{A}.y_i)_i$. Then $\mathrm{A}.y$ is contained in $\mathscr{C}$. By Conjecture 1.8, $\mathrm{A}.y$ is closed. By Theorem 1.15, for $z\in\mathrm{X}_3$ that is close enough to y, $\mathrm{A}.z$ compact implies that it can not be contained in $\mathscr{C}$. This is a contradiction.

1.10. **Ergodic decomposition.** Let μ be a Borel probability measure on X_3. We say that μ is A-ergodic iff every A-invariant Borel subset has μ-measure zero or one.

Lemma 1.25. *Let μ be a Borel probability measure on X_3. The following three are equivalent:*

(1) *μ is* A*-ergodic;*
(2) *every* A*-invariant L^1-function is constant almost everywhere;*
(3) *If $\mu=\nu_1+(1-\lambda)\nu_2$ for some $\lambda\in[0,1]$ and ν_1,ν_2 are* A*-invariant probability measure, then $\lambda=0$ or 1.*

Let $\mathrm{Prob}(\mathrm{X}_3)^{\mathrm{A}}$ be the set of A-invariant Borel probability measures on X_3 equipped with the weak-$*$ topology. And let $\mathrm{Prob}(\mathrm{X}_3)^{\mathrm{A,erg}}$ be those ergodic ones.

Theorem 1.26 (Ergodic decomposition)**.** *For every $\mu\in\mathrm{Prob}(\mathrm{X}_3)^{\mathrm{A}}$, there exists a probability measure λ_μ on $\mathrm{Prob}(\mathrm{X}_3)^{\mathrm{A}}$ with $\lambda_\mu(\mathrm{Prob}(\mathrm{X}_3)^{\mathrm{A,erg}})=1$ such that*

$$\mu=\int_{\mathrm{Prob}(\mathrm{X}_3)^{\mathrm{A}}}\nu\,\lambda_\mu(\nu).$$

More explicitly, for a compactly supported continuous function $f:\mathrm{X}\to\mathbb{R}$, let φ_f be the continuous function on $\mathrm{Prob}(\mathrm{X}_3)^{\mathrm{A}}$ defined by $\varphi(\nu)=\int f(x)\nu(x)$. Then

$$\int f(x)\mu(x)=\int_{\mathrm{Prob}(\mathrm{X}_3)^{\mathrm{A}}}\varphi_f(\nu)\,\lambda_\mu(\nu).$$

Remark 1.27. *This can be deduced from Choquet's theorem. A quick proof for the case needed can be found in* [Phe01]*.*

1.11. **Conjecture 1.10 implies Conjecture 1.8.** So take $\mathrm{A}.x$ to be a bounded A-orbit. For $T>0$, define

$$\mu_T := \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} (\mathbf{a}_{(s,t)})_*\delta_x \,\mathrm{ds}\,\mathrm{dt} \in \mathrm{Prob}(\mathrm{X}_3).$$

Since $\mathrm{A}.x$ is bounded, by passing to a subsequence, we assume $\lim_n \mu_{T_n}$ exists in $\mathrm{Prob}(\mathrm{X}_3)$. Let μ denote this limit. Then μ is A-invariant. By ergodic decomposition

$$\mu = \int_{\mathrm{Prob}(\mathrm{X}_3)^{\mathrm{A,erg}}} \nu\,\lambda_\mu(\nu).$$

Now Conjecture 1.10 says that

$$\mathrm{Prob}(\mathrm{X}_3)^{\mathrm{A,erg}} = \{\mathrm{m}_{\mathrm{A}.y},\ \mathrm{A}.y \text{ compact }\} \sqcup \{\mathrm{m}_{\mathrm{X}_3}\}.$$

As $\mathrm{m}_{\mathrm{X}_3}$ has unbounded support, λ_μ must put positive mass on certain $\mathrm{m}_{\mathrm{A}.y}$ with $\mathrm{A}.y$ compact. In particular, $\mathrm{A}.x$ contains some compact A-orbit in its closure. By Theorem 1.15, $\mathrm{A}.x$, being bounded, must be compact.

1.12. **Proof of Theorem 1.15.** Assume otherwise, namely, there exist a compact subset $\mathscr{C}\subset \mathrm{X}_3$ and a sequence $(x_n)\subset \mathrm{X}_3$ converging to $y\in\mathrm{X}_3$ such that $\mathrm{A}.x_n$ is contained in $\mathscr{C}$ for every n, $\mathrm{A}.y$ is compact and $\mathrm{A}.x_n \neq \mathrm{A}.y$ for every n.

1.12.1. *Exponential "blow-up".* Fix $\varepsilon_0>0$ such that the conclusion of Lemma 1.24 holds. For n large enough such that $x_n \in \mathcal{N}^{UA}_{\mathrm{A}.y}(0.5\varepsilon_0)$,

$$x_n = \mathbf{u}^{--}(x_n)\mathbf{u}^{++}(x_n).y(x),\quad \mathbf{u}^{--}(x_n)\in \mathrm{U}^{--}(0.5\varepsilon_0),\ \mathbf{u}^{++}(x_n)\in\mathrm{U}^{++}(0.5\varepsilon_0),\ y(x)\in \mathrm{A}.y.$$

Now we look at

$$\max\left\{\begin{array}{l} |(\mathbf{u}^{--}(x_n))_{21}|,\ |(\mathbf{u}^{--}(x_n))_{31}|,\ |(\mathbf{u}^{--}(x_n))_{32}|, \\ |(\mathbf{u}^{--}(x_n))_{12}|,\ |(\mathbf{u}^{--}(x_n))_{13}|,\ |(\mathbf{u}^{--}(x_n))_{23}| \end{array}\right\} \tag{4}$$

Without loss of generality, we are going to assume, by passing to a subsequence, that the maximum above is taken by $|(\mathbf{u}^{--}(x_n))_{12}|$ for all n and that $(\mathbf{u}^{--}(x_n))_{12}>0$ for all n.

Let

$$\beta_t := \begin{bmatrix} t & 0 & 0\\ 0 & t^{-1} & 0\\ 0 & 0 & 1\end{bmatrix}$$

Choose $t_n>0$ such that

$$t_n^2\cdot\left(\mathbf{u}^{--}(x_n)\right)_{12} = 0.5\varepsilon_0$$

Then $\beta_{t_n}.x_n$ stays inside the neighborhood $\mathcal{N}^{UA}_{\mathrm{A}.y}(0.5\varepsilon_0)$. Let $\varepsilon_n = \frac{0.5\varepsilon_0}{M_n^2}$ be the maximum appearing in Equa.(4). About the size of $(\mathbf{u}^{--}(\beta_{t_n}.x_n))_{ij}$ $(i\neq j)$, we have

$$\begin{bmatrix} & 0.5\varepsilon_0 & \leq \frac{0.5\varepsilon_0}{M_n}\\ \leq \frac{0.5\varepsilon_0}{M_n^4} & & \leq\frac{0.5\varepsilon_0}{M_n^3}\\ \leq\frac{0.5\varepsilon_0}{M_n^3} & \leq\frac{0.5\varepsilon_0}{M_n} & \end{bmatrix}$$

By passing to a further subsequence, assume $\beta_{t_n}.x_n$ converges to x_∞ and $\beta_{t_n}.y(x_n)$ converges to y_∞. Then we have

$$x_\infty = u_{12}(0.5\varepsilon_0).y_\infty.$$

By definition, $\mathrm{A}.x_\infty$ is contained in $\mathscr{C}$.

1.12.2. *Promotion.* Now we use a one-parameter subgroup of A that commutes with $u_{12}(\mathbb{R})$. Define

$$\alpha_t := \begin{bmatrix} t & 0 & 0\\ 0 & t & 0\\ 0 & 0 & t^{-2}\end{bmatrix}$$

Then

$$\alpha_t.x_\infty = u_{12}(0.5\varepsilon_0)\alpha_t.y_\infty. \tag{5}$$

Lemma 1.28. *$\{\alpha_t.y_\infty,\ t\in\mathbb{R}\}$ is dense in* $\mathrm{A}.y_\infty = \mathrm{A}.y$.

By Lemma 1.28 and Equa.(5),

$$\overline{\mathrm{A}.x_\infty} \supset u_{12}(0.5\varepsilon_0)\mathrm{A}.y$$

Using the A-invariance of the LHS, we get

$$\overline{\mathrm{A}.x_\infty} \supset u_{12}(\mathbb{R}^+)\mathrm{A}.y.$$

To get a contradiction, it suffices to show that $u_{12}(\mathbb{R}^+)\mathrm{A}.y$ (as lattices) contains arbitrarily small non-zero vectors.

Given $\varepsilon > 0$, one can find $(u,v,w)^{\mathrm{tr}} \in y$ with $u < 0$, $v > 0$. Take $t > 0$ large enough such that $|e^{-t}v| < \varepsilon$ and $|e^{-t}w| < \varepsilon$. Then take $r := \frac{e^{2t}u}{-e^{-t}v}$. One has:

$$\begin{bmatrix} 1 & r & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-t}v \\ e^{-t}w \end{bmatrix},$$

which is a vector contained in the lattice $u_{12}(r)a.y$ for some $a \in \mathrm{A}$ and $r > 0$. This shows that the sys$(\cdot)$ of elements in $\overline{\mathrm{A}.x_\infty}$ could tend to 0. By the continuity of sys$(\cdot)$, $\overline{\mathrm{A}.x_\infty}$ is non-compact, a contradiction.

1.13. **Littlewood conjecture for cubic numbers.** Using a variant of the isolation principle presented above, one can show that

Theorem 1.29. *Let K be a cubic totally real number field and $\alpha, \beta \in K$. Then (α,β) satisfies the Littlewood conjecture.*

By taking transpose inverse $(\cdot)^{-\mathrm{tr}}$, one sees that $\mathrm{A}^+.\Lambda_{\alpha,\beta}$ is unbounded iff $\mathrm{A}^-.\Lambda'_{\alpha,\beta}$ is unbounded where

$$\Lambda'_{\alpha,\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\alpha & -\beta & 1 \end{bmatrix}.\mathbb{Z}^3.$$

Let $\{\sigma_1 = \mathrm{id}, \sigma_2, \sigma_3\}$ denote the three different embedding of $K \hookrightarrow \mathbb{R}$. Let

$$M_0 := \begin{bmatrix} -\sigma_3(\alpha) & -\sigma_3(\beta) & 1 \\ -\sigma_2(\alpha) & -\sigma_2(\beta) & 1 \\ -\alpha & -\beta & 1 \end{bmatrix}$$

Let $\lambda_0 \in \mathbb{R}$ such that $\det(\lambda_0 \cdot M_0) = 1$.

Lemma 1.30. *$\mathrm{A}.(\lambda_0 M_0).\mathbb{Z}^3$ is compact.*

Proof. Dirichlet's unit theorem and commensurability of lattices. □

Note that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\alpha & -\beta & 1 \end{bmatrix} = \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ u_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \cdot M_0$$

for some real numbers t_i, u_{ij}. Thus

$$\alpha_s.\Lambda'_{\alpha,\beta} = \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ u_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & u_{12} & s^3u_{13} \\ 0 & 1 & s^3u_{23} \\ 0 & 0 & 1 \end{bmatrix} \cdot \alpha_s M_0.\mathbb{Z}^3$$

Take some sequence $s_n \to 0$ such that $\lim \alpha_{s_n}(\lambda_0 M_0).\mathbb{Z}^3$ exists)and is equal to y_1. Then

$$x_1 := \lim \alpha_{s_n}\Lambda'_{\alpha,\beta} = \begin{bmatrix} t'_1 & 0 & 0 \\ 0 & t'_2 & 0 \\ 0 & 0 & t'_3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ u_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & u_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .y_1$$

Using α_s again,

$$\alpha_s.x_1 = \begin{bmatrix} t'_1 & 0 & 0 \\ 0 & t'_2 & 0 \\ 0 & 0 & t'_3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ u_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & u_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \alpha_s.y_1$$

By Lemma 1.28,

$$\overline{\{\alpha_s.x_1,\ s\in\mathbb{R}_{<0}\}} \supset \begin{bmatrix} t_1' & 0 & 0 \\ 0 & t_2' & 0 \\ 0 & 0 & t_3' \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ u_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & u_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathrm{A}.y_1$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ u_{21}' & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & u_{12}' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathrm{A}.y_1$$

Therefore

$$\overline{\mathrm{A}^-.\Lambda'_{\alpha,\beta}} \supset \left\{ \begin{bmatrix} 1 & 0 & 0 \\ su_{21}' & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & s^{-1}u_{12}' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathrm{A}.y_1 \,\middle|\, s\in\mathbb{R}^+ \right\}$$

Note that u_{12} and hence u_{12}' is non-zero. Thus, for non-zero $(l,m,n)^{\mathrm{tr}} \in \mathrm{A}.y_1$ (certainly $l \neq 0$!), by taking

$$s := \frac{mu_{12}}{l}$$

we get

$$\begin{bmatrix} 0 \\ mu_{12}u_{21} + m + mu_{21}u_{12} \\ n \end{bmatrix} \in \overline{\mathrm{A}^-.\Lambda'_{\alpha,\beta}}$$

Now we choose $(l,m,n) \in \mathrm{A}.y_1$ such that

$$l<0,\ mu_{12}>0,\ m,n \text{ very small}$$

Then invoke the A^--action on such a vector. This shows that $\mathrm{sys}(\mathrm{A}^-.\Lambda'_{\alpha,\beta})$ can not be bounded away from 0.

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