LECTURE 3

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Contents

Notation

1. Lecture 3, Hausdorff dimension of BAD

Reference: See [Mat95] for more on Hausdorff dimensions. The main result in this lecture follows [KM96]. Lemma 1.3 is taken from [McM87].

1.1. **Prelude.** When a set has Lebesgue measure zero, there is a more refined way of measuring its size: Hausdorff dimension. A Lebesgue-null subset of $[0,1)$ could have dimension from 0 to 1. The classical Cantor's middle third set has Hausdorff dimension $\frac{\log 2}{\log 3}$. In this lecture we will show that the set of badly approximable numbers, which is small in terms of Lebesgue measure, is big in terms of Hausdorff dimension. Its Hausdorff dimension is equal to 1, proved by Jarnik. We are going to follow the proof by Kleinbock–Margulis, using the mixing property of geodesic flow to construct a Cantor-like set in **BAD** with large Hausdorff dimension.

1.2. **Hausdorff dimension.** Let $E \subset [0,1)$. For $s > 0$ and $\varepsilon > 0$, define

$$\mathcal{H}^s_\varepsilon := \inf \left\{ \sum \mathrm{diam}(I_i)^s \;\middle|\; E \subset \bigcup I_i \text{ countable union of intervals, } \mathrm{diam}(I_i) < \varepsilon,\ \forall\, i \right\}.$$

For $s > 0$, define

$$\mathcal{H}^s(E) := \lim_{\varepsilon \to 0} \mathcal{H}^s_\varepsilon(E).$$

Note that such a limit indeed exists (possibly $+\infty$) since $\mathcal{H}^s_\varepsilon(E)$ is non-decreasing as ε decreases to 0.

The Hausdorff dimension is defined by

$$\dim_{\mathrm{H}}(E) := \inf \{ s \geq 0 \mid \mathcal{H}^s(E) = 0 \}. \tag{1}$$

If non-empty (otws, $\dim_{\mathrm{H}}(E) = 0$.), then one can directly check that

$$\dim_{\mathrm{H}}(E) := \sup \{ s \geq 0 \mid \mathcal{H}^s(E) = +\infty \}.$$

From the definition, one sees that

Lemma 1.1. *Let $\alpha \in [0,1]$ and E be a subset of $[0,1)$. If there exist $C, \varepsilon > 0$ such that for every covering of E by countably many intervals (I_i) with $\mathrm{diam}(I_i) < \varepsilon$, one has $\sum \mathrm{diam}(I_i)^\alpha > C$, then $\dim_{\mathrm{H}}(E) \geq \alpha$.*

The main goal of this lecture is to prove that

Theorem 1.2. *The Hausdorff dimension of* **BAD** *is equal to* 1.

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1.3. **Lower bound of Hausdorff dimension.** For $N \in \mathbb{Z}^+$, let $\mathscr{I}_N$ denote the collection of intervals

$$\mathscr{I}_N := \left\{ [\frac{i}{N}, \frac{i+1}{N}) \,\middle|\, i = 0, 1, ..., N-1 \right\}.$$

Lemma 1.3. *Fix $N \in \mathbb{Z}^+$ and $\delta \in (0,1)$. Suppose that for each $k \in \mathbb{Z}^+$, we have a subset $\mathscr{E}_k$ of $\mathscr{I}_{N^k}$ (by default, also set $\mathscr{E}_0 := \{[0,1)\}$) satisfying*

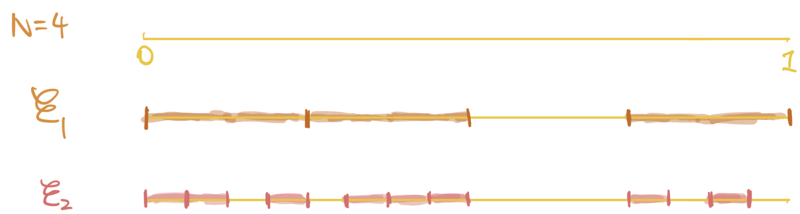
(1) *for every $k \in \mathbb{Z}^+$ and $E \in \mathscr{E}_k$ there exists $E' \in \mathscr{E}_{k-1}$ containing E;*
(2) *for every $k \in \mathbb{Z}^+$ and $E \in \mathscr{E}_{k-1}$,*

$$\frac{\#\{F \in \mathscr{E}_k \mid F \subset E\}}{N} \geq 1 - \delta.$$

Let $E_\infty := \bigcap_{k=1}^{\infty} \bigcup_{E \in \mathscr{E}_k} E$. Then

$$\dim_{\mathrm{H}}(E_\infty) \geq 1 - \frac{\log((1-\delta)^{-1})}{\log N}.$$

Here is a picture of Cantor like sets...



1.4. **Convergence of measures.** We claim that the "natural" probability measures supported on $\cup_{E \in \mathscr{E}_k} E$ converges as $k \to +\infty$ under the weak$*$ topology[1]. By Riesz's representation theorem (See Rudin's book, real and complex analysis, Theorem 2.14.), we may and do specify a measure by integrating compactly supported continuous functions.

Let f be a continuous function on $[0,1]$, define

$$L_1(f) := \frac{1}{\#\mathscr{E}_1} \sum_{E_1 \in \mathscr{E}_1} N \cdot \int_{E_1} f(x)\mathrm{d}x.$$

So this is integrating f against the normalized probability measure supported on $\bigsqcup_{E \in \mathscr{E}_1} E$. Then one "refines" this measure by

$$L_2(f) := \frac{1}{\#\mathscr{E}_1} \sum_{E_1 \in \mathscr{E}_1} \frac{1}{\#\{E_2 \in \mathscr{E}_2, E_2 \subset E_1\}} \sum_{E_2 \in \mathscr{E}_2, E_2 \subset E_1} N^2 \cdot \int_{E_2} f(x)\mathrm{d}x.$$

In general, for $E \in \mathscr{E}_k$, let

$$\mathscr{E}_{k+1}(E) := \{F \in \mathscr{E}_{k+1} \mid F \subset E\}.$$

Also, given $k \in \mathbb{Z}^+$ and $E \in \mathscr{E}_k$, define

$$E =: E^k \subset E^{k-1} \subset ... \subset E^1 \subset E^0 := [0,1) \tag{2}$$

by requiring E_{i-1} to be the unique element in $\mathscr{E}_{i-1}$ containing $E_i \in \mathscr{E}_i$ for $i = k, k-1, ..., 1$. Further define the weight for E by

$$\mathbf{w}_E := \frac{1}{\#\mathscr{E}_k(E^{k-1}) \cdot \#\mathscr{E}_{k-1}(E^{k-2}) \cdot ... \cdot \#\mathscr{E}_2(E^1) \cdot \#\mathscr{E}_1}.$$

Now for general $k \in \mathbb{Z}^+$, a positive linear functional L_k is defined for $f \in C[0,1]$[2] by

$$L_k(f) := \sum_{E \in \mathscr{E}_k} \mathbf{w}_E \cdot N^k \cdot \int_E f(x)\mathrm{d}x.$$

Using the fact that f's are uniformly continuous, one can check that

[1] A sequence of measures (μ_n) converges to μ under the **weak$*$ topology** iff $\int f(x)\,\mu_n(x)$ converges to $\int f(x)\,\mu(x)$ for every continuous function f on $[0,1]$.

[2] For a topological space X, let $C(X)$ denote the Banach space of continuous functions on X equipped with the sup-norm.

Lemma 1.4. *For every $f \in C[0,1]$, the limit $L_\infty(f) := \lim_{k\to\infty} L_k(f)$ exists and $f \mapsto L_\infty(f)$ is a bounded positive linear functional on $C[0,1]$ mapping the constant one function to 1. Consequently, there exists a probability measure μ such that $L_\infty(f) = \int_0^1 f(x)\mu(x)$.*

We reserve μ for such a measure till the end of the proof of Lemma 1.3.

1.5. **Proof of Lemma 1.3.** For $E \in \mathscr{E}_k$, find $E =: E^k \subset E^{k-1} \subset ... \subset E^1 \subset E^0 := [0,1)$ as in Equa.(2). Then $\mu(E) = \mathbf{w}_E$. By assumption, each $\#\mathscr{E}_i(E^{i-1}) \geq (1-\delta)N$. Therefore,

$$\mu(E) \leq \frac{1}{(1-\delta)^k N^k}.$$

Fix a covering of $E \subset \bigcup I_i$ by countably many intervals. For each i, let k_i be the unique positive integer such that

$$\frac{1}{N^{k_i+1}} \leq \operatorname{diam}(I_i) < \frac{1}{N^{k_i}}.$$

Let $\mathscr{E}_{k_i}(I_i)$ collects intervals E in $\mathscr{E}_{k_i}$ with $E \cap I_i \neq \emptyset$. We note that

Lemma 1.5. $\#\mathscr{E}_{k_i}(I_i) \leq 2$.

On the other hand, for $E \in \mathscr{E}_{k_i}(I_i)$ and $\alpha \in [0,1]$,

$$\mu(E) \leq \frac{N}{(1-\delta)^{k_i} N^{k_i+1}} \leq \frac{N \operatorname{diam}(I_i)^\alpha}{(1-\delta)^{k_i} N^{(k_i+1)(1-\alpha)}} = \frac{1}{((1-\delta)N^{1-\alpha})^{k_i}} N^\alpha \operatorname{diam}(I_i)^\alpha,$$

which is $\leq N^\alpha \operatorname{diam}(I_i)^\alpha$ provided $(1-\delta)N^{1-\alpha} \geq 1$, or equivalently,

$$\alpha \leq 1 - \frac{\log((1-\delta)^{-1})}{\log N}.$$

Therefore, for α satisfying the inequality above,

$$1 = \mu(E) \leq \sum \mu(I_i) \leq \sum_i \sum_{E \in \mathscr{E}_{k_i}(I_i)} \mu(E) \leq \sum_i 2 \cdot N^\alpha \operatorname{diam}(I_i)^\alpha$$

$$\implies \sum \operatorname{diam}(I_i)^\alpha \geq \frac{1}{2N^\alpha}.$$

As $0.5N^{-\alpha}$ is a positive constant independent of the covering (I_i) chosen, this completes the proof of Lemma 1.3 by Lemma 1.1.

1.6. **A remark.** One could have rewritten the above proof into two steps (let $\alpha < 1 - \frac{\log((1-\delta)^{-1})}{\log N}$):

(1) Construct a probability measure μ on E_∞ with the property that for some $C > 0$, for every r small enough, $\mu((x-r, x+r)) < Cr^\alpha$ holds for μ almost all x;
(2) Show that whenever a set has positive μ-measure, it must has Hausdorff dimension at least α.

It turns out that the second step has a converse to it. Namely, if a set E has Hausdorff dimension $> \alpha$, then one can find a probability measure μ supported on E (meaning, $\mu(E) = 1$) such that for some $C > 0$, for every r small enough, $\mu((x-r, x+r)) < Cr^\alpha$ holds for μ almost all x. This is called the Frostman Lemma.

1.7. **Construct Cantor-like sets in BAD.** We give a construction of $(\mathscr{E}_k)$, whose intersections E_∞ lies inside **BAD**. For this we start with a positive integer N and a compact subset $\mathscr{C}$ of X_2. Define $\mathscr{C}' := \mathbf{u}^+_{[-1,1]}.\mathscr{C}$.

For $x \in \mathrm{X}_2$ and $I = [a_I, b_I) \in \mathscr{I}_N$, define

$$\phi_I(x) := \mathbf{a}_{\frac{1}{2}\log N} \mathbf{u}^+_{a_I}.x = \begin{bmatrix} N^{\frac{1}{2}} & 0 \\ 0 & N^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} 1 & a_I \\ 0 & 1 \end{bmatrix}.x.$$

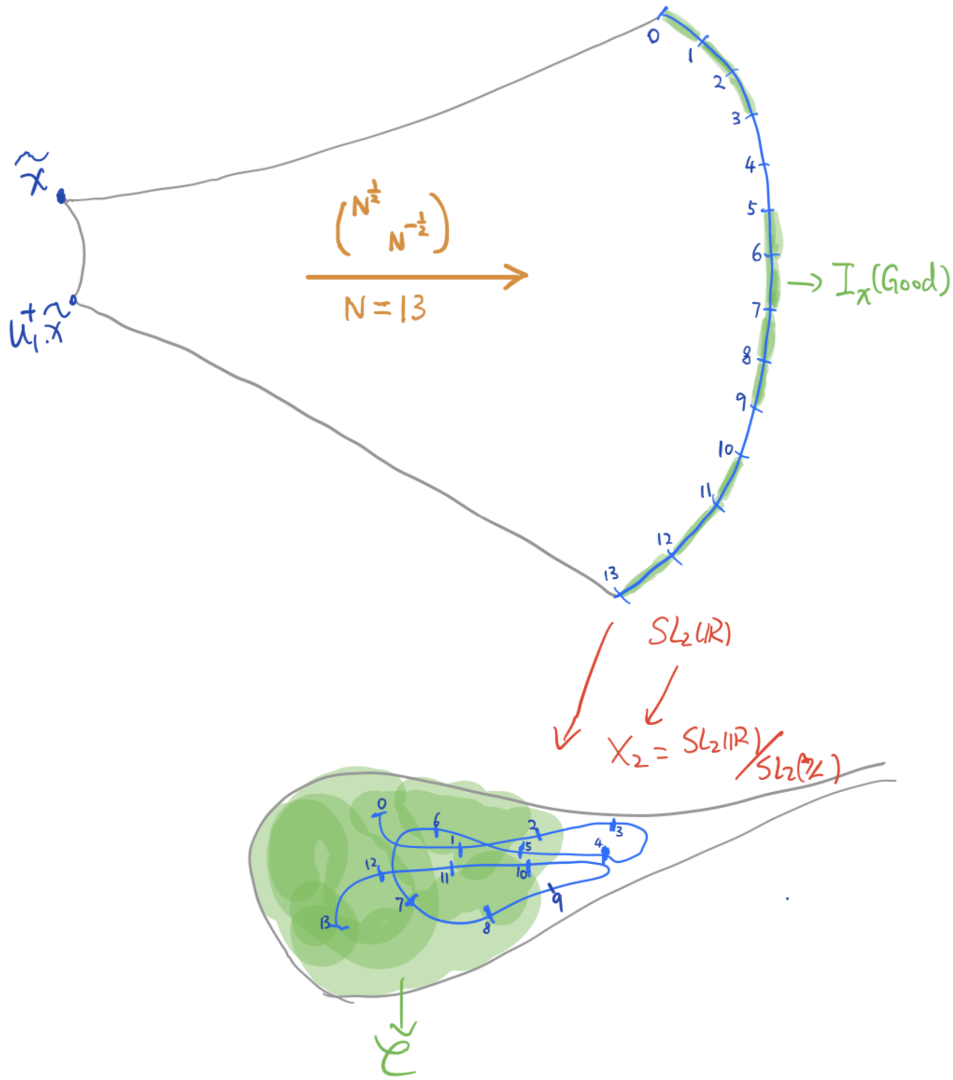
Hence

$$\mathbf{a}_{\frac{1}{2}\log N} \mathbf{u}^+_I.x = \mathbf{u}^+_{[0,1)}.\phi_I(x).$$

For $x \in \mathrm{X}_2$, define a subset $\mathscr{I}_x(\mathrm{Good}) \subset \mathscr{I}_N$ by

$$I \in \mathscr{I}_x(\mathrm{Good}) \iff \mathbf{a}_{\frac{1}{2}\log N} \mathbf{u}^+_I.x \cap \mathscr{C} \neq \emptyset \iff \mathbf{u}^+_{[0,1)}.\phi_I(x) \cap \mathscr{C} \neq \emptyset.$$

Thus $\phi_I(x) \in \mathscr{C}'$ if $I \in \mathscr{I}_x(\mathrm{Good})$.



1.7.1. *Initial steps.* Let $x_0 := \mathbb{Z}^2$, we define $\mathscr{E}_1 := \mathscr{I}_{x_0}(\mathrm{Good})$.

Let

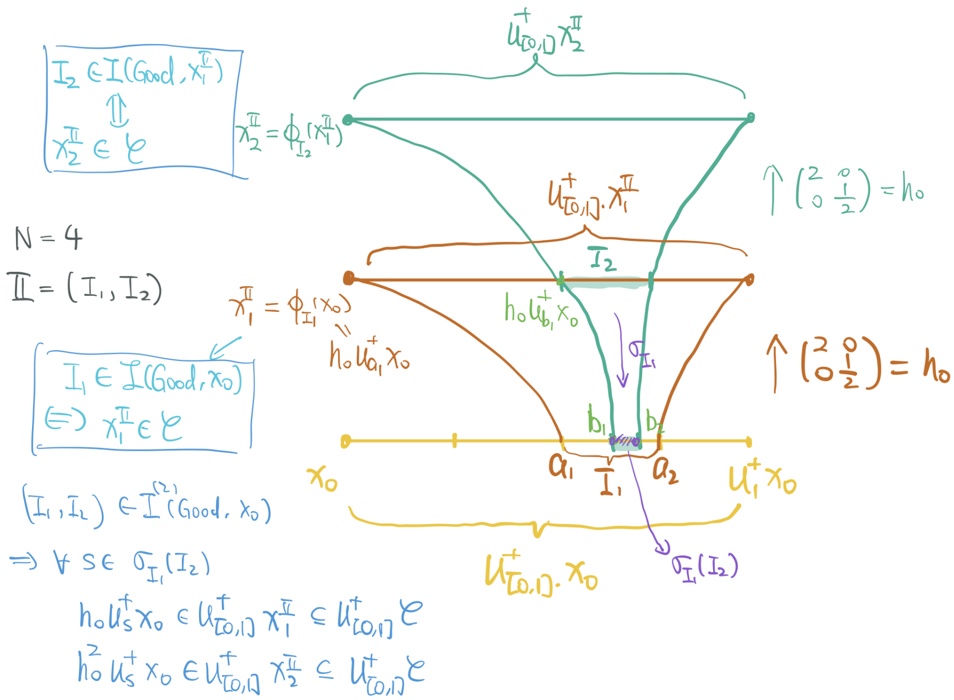
$$\mathscr{I}^2_{x_0}(\mathrm{Good}) := \left\{ (I_1, I_2) \;\middle|\; I_1 \in \mathscr{I}_{x_0}(\mathrm{Good}),\ I_2 \in \mathscr{I}_{\phi_{I_1}(x_0)}(\mathrm{Good}) \right\}$$

For an interval $I = [a_I, b_I)$, let σ_I be the unique (orientation-preserving) affine transformation sending $[0,1)$ to $[a_I, b_I)$, namely,

$$\begin{aligned} \sigma_I : [0,1) &\to I = [a_I, b_I) \\ t &\mapsto t a_I + (1-t) b_I. \end{aligned}$$

Define

$$\mathscr{E}_2 := \left\{ \sigma_{I_1}(I_2) \;\middle|\; (I_1, I_2) \in \mathscr{I}^2_{x_0}(\mathrm{Good}) \right\}.$$



1.7.2. *In general....* Given a finite sequence $\mathbb{I} := (I_1, I_2, ..., I_k)$ of elements in $\mathscr{I}_N$, for $i \in \{1, ..., k\}$, let

$$x_i^{\mathbb{I}} := \phi_{I_i} \circ \phi_{I_{i-1}} \circ ... \circ \phi_{I_1}(x_0).$$

By default, set $x_0^{\mathbb{I}} := x_0$. Define, for $k \in \mathbb{Z}^+$,

$$\mathscr{I}^k_{x_0}(\text{Good}) := \left\{ \mathbb{I} = (I_1, ..., I_k) \;\middle|\; I_i \in \mathscr{I}_{x^{\mathbb{I}}_{i-1}}(\text{Good}),\ \forall\, i = 1, ..., N \right\},$$

and

$$\mathscr{E}_k := \left\{ \sigma_{I_1} \circ ... \circ \sigma_{I_{k-1}}(I_k) \;\middle|\; (I_1, ..., I_k) \in \mathscr{I}^k_{x_0}(\text{Good}) \right\}.$$

Lemma 1.6. *Given $N \in \mathbb{Z}^+$ and a compact subset $\mathscr{C}$ of X_2. The family of sets $(\mathscr{E}_k)$ constructed above satisfy condition (1) in Lemma 1.3. And $E_\infty := \cap_{k=1}^{\infty} \cup_{E \in \mathscr{E}_k} E$ is contained in* **BAD**.

Proof. The first part follows from the construction. Turn to the second part. By definition, for $s \in \mathscr{E}_k$ associated with $\mathbb{I} = (I_1, ..., I_k)$, and for each $i = 1, .., k$, $x_i^{\mathbb{I}}$ is contained in $\mathscr{C}$. Also, $\mathbf{a}^i_{\frac{1}{2}\log N} \mathbf{u}^+_s . x_0 \subset \mathbf{u}^+_{[0,1]} . x_i^{\mathbb{I}}$. Thus $\mathbf{a}^k_{\frac{1}{2}\log N} \mathbf{u}^+_s . x_0 \subset \mathscr{C}'$.

Now take $s \in E_\infty$, we have seen that $\mathbf{a}^k_{\frac{1}{2}\log N} \mathbf{u}^+_s . x_0 \subset \mathscr{C}'$ for every $k \in \mathbb{Z}^+$. Thus $\mathbf{a}_{\geq 0} \mathbf{u}^+_s . x_0 \subset \mathbf{a}_{[0, \frac{1}{2}\log N]} . \mathscr{C}'$ is bounded. By Dani correspondence, s is badly approximable. □

Note that actually **BAD** can be written as a countable union of such E_∞'s. And we need to choose $N, \mathscr{C}$ such that condition (2) from Lemma 1.3 holds for δ fixed but N tends to infinity.

1.8. **Consequence of mixing.** Recall from lecture 2.5 we have

Theorem 1.7. *Fix $y_0 \in \mathrm{X}_2$, $\varepsilon_0 \in (0,1)$ and a compact subset $\mathscr{C}$ of X_2. There exist $\delta, T > 0$ and $M \in 2\mathbb{Z}^+$ such that for every $x \in \mathscr{C}$ and $T' > T$,*

$$\int_{-0.5}^{0.5} \mathbf{1}_{B_{y_0}(\varepsilon_0)}(\mathbf{a}_{T'} \mathbf{u}^+_t . x)\, \mathrm{dt} > \delta.$$

Fix some $y_0 \in \mathrm{X}_2$ and $\varepsilon_0 \in (0,1)$. Let $\mathscr{C}_3 := \mathbf{u}^+_{[-3,3]} . \overline{B_{y_0}(\varepsilon_0)}$. By Theorem 1.7, we find $\delta_0 > 0$, $T_0 > 0$ such that for every $T > T_0$ and $x \in \mathscr{C}_3$,

$$\text{Leb}\left\{ t \in [-0.5, 0.5] \;\middle|\; \mathbf{a}_T \mathbf{u}^+_t . x \in B_{y_0}(\varepsilon_0) \right\} > \delta_0.$$

For $x \in \mathscr{C}_2 := \mathbf{u}^+_{[-2,2]}.\overline{B_{y_0}(\varepsilon_0)}$, apply the above to $\mathbf{u}^+_{0.5}.x \in \mathscr{C}_3$, we get

$$\mathrm{Leb}\left\{t \in [0,1] \;\middle|\; \mathbf{a}_T \mathbf{u}^+_t.x \in B_{y_0}(\varepsilon_0)\right\} > \delta_0, \quad \forall T > T_0. \tag{3}$$

Apply the Cantor-like set construction to N with $\frac{1}{2}\log(N) > T_0$ and $\mathscr{C} := \mathscr{C}_1 := \mathbf{u}^+_{[-1,1]}.\overline{B_{y_0}(\varepsilon_0)}$. For simplicity write $h_N := \mathbf{a}_{\frac{1}{2}\log(N)}$.

Take $E \in \mathscr{E}_k$, we need to bound

$$\frac{\#\left\{F \in \mathscr{E}_{k+1} \mid F \subset E\right\}}{N}$$

from below. Recall that E is of the form $\sigma_{I_1} \circ ... \circ \sigma_{I_{k-1}}(I_k)$ for some $\mathbb{I} = (I_i)_{i=1}^k \subset \mathscr{I}_N$. And if $I_i = [a_i, b_i)$, we have defined

$$\mathscr{C}_1 \ni x_k^{\mathbb{I}} = h_N \mathbf{u}^+_{a_k}.x^{\mathbb{I}}_{k-1} = ... = (h_N \mathbf{u}^+_{a_k}) \cdot (h_N \mathbf{u}^+_{a_{k-1}}) \cdot ... \cdot (h_N \mathbf{u}^+_{a_1}).x_0$$

where $x_0 = \mathbb{Z}^2$ is the identity coset. Moreover, we have a bijection

$$\mathscr{I}_{x_k^{\mathbb{I}}}(\mathrm{Good}) \to \{F \in \mathscr{E}_{k+1} \mid F \subset E\}$$
$$I \mapsto \sigma_{I_1} \circ ... \circ \sigma_{I_k}(I).$$

Recall an interval $I = [a_I, b_I) \in \mathscr{I}_N$ is contained in $\mathscr{I}_{x_k^{\mathbb{I}}}(\mathrm{Good})$ iff $h_N \mathbf{u}^+_{a_I}.x_k^{\mathbb{I}} \in \mathscr{C}_1$. As

$$\begin{aligned} I \notin \mathscr{I}_{x_k^{\mathbb{I}}}(\mathrm{Good}) &\implies h_N \mathbf{u}^+_{a_I}.x_k^{\mathbb{I}} \notin \mathscr{C}_1 \\ &\implies \mathbf{u}^+_{[0,1]} h_N \mathbf{u}^+_{a_I}.x_k^{\mathbb{I}} \cap B_{y_0}(\varepsilon_0) = \emptyset \\ &\iff h_N \mathbf{u}^+_{I}.x_k^{\mathbb{I}} \cap B_{y_0}(\varepsilon_0) = \emptyset. \end{aligned}$$

Thus,

$$\mathrm{Leb}\left\{t \in [0,1] \;\middle|\; \mathbf{a}_T \mathbf{u}^+_t.x \notin B_{y_0}(\varepsilon_0)\right\} > 1 - \frac{\#\mathscr{I}_{x_k^{\mathbb{I}}}(\mathrm{Good})}{N}.$$

Combined with Equa.(3) (note that $\frac{1}{2}\log(N) > T_0$),

$$\begin{aligned} 1 - \delta_0 &> 1 - \frac{\#\mathscr{I}_{x_k^{\mathbb{I}}}(\mathrm{Good})}{N} \\ \implies \frac{\#\left\{F \in \mathscr{E}_{k+1} \mid F \subset E\right\}}{N} &= \frac{\#\mathscr{I}_{x_k^{\mathbb{I}}}(\mathrm{Good})}{N} > \delta_0. \end{aligned}$$

By Lemma 1.3 and Lemma 1.6, we have

$$\dim_H(\mathbf{BAD}) \geq \dim_H(E_\infty) \geq 1 - \frac{\log(\delta_0^{-1})}{\log(N)}.$$

Letting $N \to +\infty$, we get

$$\dim_H(\mathbf{BAD}) \geq 1.$$

Remark 1.8. *You can also show that* $\dim_H(E_\infty)$ *is strictly smaller than* 1.

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