LECTURE 2

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Contents

Notation

Vectors in $\mathbb{R}^n$, by default, are written as column vectors. For a few $\mathbf{x}_1, ..., \mathbf{x}_k$, write $(\mathbf{x}_1, ..., \mathbf{x}_k)$ for the n-by-k matrix whose i-th column is given by $\mathbf{x}_i$. We use I_2 to denote the two-by-two identity matrix.

1. Lecture 2, Space of lattices of $\mathbb{R}^2$, Dani's correspondence and Ergodic theory

One may consult Cassels' book [Cas59] for facts about lattices in $\mathbb{R}^n$. For an introduction to ergodic theory, we recommend Einsiedler–Ward's book [EW11]. The proof of mixing of the geodesic flow is taken from Witte Morris' excellent book [Mor15]. For relation between Khintchine's theorem and exponential mixing, which is not discussed here, see the work of Kleinbock–Margulis [KM99]. The interaction between homogeneous dynamics and Diophantine approximation (especially the metric aspects) is very fruitful. See [Kle23] for a survey.

1.1. Prelude.

Certain problems in Diophantine approximations can be restated in terms of lattices in $\mathbb{R}^n$ (the study of such objects is called "geometry of numbers"). Rather than studying individual lattices one-by-one, it is fruitful to study all lattices at the same time. It turns out that this space allows the transitive action of a linear group. Hence tools from linear algebra can be applied. Moreover, this (non-compact) space has a finite invariant measure. Therefore, tools from ergodic theory kick in.

Towards the end of this lecture, we will provide an alternative proof of **BAD** having zero Lebesgue measure from this point of view.

1.2. Unimodular lattices in $\mathbb{R}^2$.

Definition 1.1. *A discrete subgroup $\Lambda \leq \mathbb{R}^2$ is said to be a* ***lattice*** *iff there exists linearly independent $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ such that $\Lambda = \mathbb{Z}\mathbf{v} + \mathbb{Z}\mathbf{w}$. The co-volume of a lattice, denoted as $\|\Lambda\|$, is defined to be $|\det(\mathbf{v}, \mathbf{w})| = \|\mathbf{v} \wedge \mathbf{w}\|$. A lattice is said to be* ***unimodular*** *iff its co-volume is equal to one.*

Definition 1.2. *Let X_2 denote the set of all unimodular lattices in $\mathbb{R}^2$.*

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Lemma 1.3. *Let Λ be a lattice of $\mathbb{R}^2$ and $F \subset \mathbb{R}^2$ be a Borel subset. If $F \cap (F + \mathbf{v}) = \emptyset$ for every nonzero $\mathbf{v} \in \Lambda$, then $\mathrm{Leb}(F) \leq \|\Lambda\|$. On the other hand, if $\mathbb{R}^2 = \bigcup_{\mathbf{v}\in\Lambda} F + \mathbf{v}$, then $\mathrm{Leb}(F) \geq \|\Lambda\|$.*

If both conditions are met, we call F a **strict fundamental domain** of Λ.

Proof. Note that there exists a strict fundamental domain F_0 for Λ with $\mathrm{Leb}(F_0) = \|\Lambda\|$. For instance, if $\Lambda = \mathbb{Z}\mathbf{v} + \mathbb{Z}\mathbf{w}$, then F_0 can be taken to be $[0,1)\mathbf{v} + [0,1)\mathbf{w}$. Let $F_{\mathbf{v}} := F \cap (F_0 - \mathbf{v})$ for $\mathbf{v} \in \Lambda$. Then $F = \bigsqcup_{\mathbf{v}\in\Lambda} F_{\mathbf{v}}$ and hence $\mathrm{Leb}(F) = \sum \mathrm{Leb}(F_{\mathbf{v}}) = \sum \mathrm{Leb}(F_{\mathbf{v}} + \mathbf{v})$.

First assume $F \cap (F + \mathbf{v}) = \emptyset$ for every nonzero $\mathbf{v} \in \Lambda$. Then $(F_{\mathbf{v}} + \mathbf{v})_{\mathbf{v}\in\Lambda}$ are disjoint from each other since $(F + \mathbf{v})$'s are. So

$$\mathrm{Leb}(F) = \sum \mathrm{Leb}(F_{\mathbf{v}} + \mathbf{v}) \leq \mathrm{Leb}(F_0) = 1.$$

Next assume $\mathbb{R}^2 = \bigcup_{\mathbf{v}\in\Lambda} F + \mathbf{v}$. Then $\bigcup F_{\mathbf{v}} + \mathbf{v} = F$. Thus

$$\mathrm{Leb}(F) = \sum \mathrm{Leb}(F_{\mathbf{v}} + \mathbf{v}) \geq \mathrm{Leb}(F_0) = 1.$$

□

We equip X_2 with the following topology: A subset $U \subset \mathrm{X}_2$ is open iff for every $\Lambda \in U$, say $\Lambda = \mathbb{Z}\mathbf{v} + \mathbb{Z}\mathbf{w}$, there exists $\varepsilon > 0$ such that every unimodular lattice $\Lambda' = \mathbb{Z}\mathbf{v}' + \mathbb{Z}\mathbf{w}'$ with $\|\mathbf{v} - \mathbf{v}'\| < \varepsilon$, $\|\mathbf{w} - \mathbf{w}'\| < \varepsilon$ belongs to U. Equivalently, we equip X_2 with the Chabauty topology.

Lemma 1.4. X_2 *is a separable metrizable space.*

There are different ways of showing X_2 is metrizable. For instance, one can show that for every $\Lambda = \mathbb{Z}\mathbf{v} + \mathbb{Z}\mathbf{w} \in \mathrm{X}_2$ and every $\varepsilon > 0$, there exists $\Lambda' = \mathbb{Z}\mathbf{v}' + \mathbb{Z}\mathbf{w}' \in \mathrm{X}_2$ with $\|\mathbf{v}' - \mathbf{v}\|, \|\mathbf{w}' - \mathbf{w}\| < \varepsilon$ and $\mathbf{v}', \mathbf{w}' \in \mathbb{Q}^2$. This would imply that X_2 is regular (every x and every neighborhood $\mathcal{N}$ of x, there exists a smaller one whose closure is contained in $\mathcal{N}$) and has a countable basis (countably many open subsets that 1. cover X_2, and 2.any intersection of two containing some x contains a third one containing the same x). Then invoke Urysohn's metrization theorem.

Note that there exist distinct $\Lambda, \Lambda' \in \mathrm{X}_2$ such that for every $\varepsilon > 0$, there exist $\mathbf{v}, \mathbf{w}, \mathbf{w}, \mathbf{w}'$ with $\Lambda = \mathbb{Z}\mathbf{v} + \mathbb{Z}\mathbf{w}$ and $\mathbf{v}', \mathbf{w}' \in \Lambda'$ such that $\|\mathbf{v} - \mathbf{v}'\| < \varepsilon$, $\|\mathbf{w} - \mathbf{w}'\| < \varepsilon$. However, it is not clear to me whether one can further require $\mathbf{v}', \mathbf{w}'$ to form a $\mathbb{Z}$-basis of Λ'.

1.3. **Systole function and Mahler's criterion.**

Definition 1.5. *For a lattice Λ, let $\mathrm{sys}(\Lambda) := \inf_{\mathbf{v}\neq\mathbf{0}\in\Lambda} \|\mathbf{v}\|$.*

Lemma 1.6. $\mathrm{sys} : \mathrm{X}_2 \to \mathbb{R}_{>0}$ *is a continuous function.*

Theorem 1.7. $\mathrm{sys} : \mathrm{X}_2 \to \mathbb{R}_{>0}$ *is a bounded proper continuous function.*

Proof. It suffices to show that, given $c_0 > 0$, for every sequence $(\Lambda_n) \subset \mathrm{X}_2$ with $\mathrm{sys}(\Lambda_n) > c_0$ for all n, there exists a convergent subsequence. It suffices, for every $\Lambda \in \mathrm{X}_2$ with $\mathrm{sys}(\Lambda) > c_0$, to find a constant $C > 1$ (depending on c_0) such that $\Lambda = \mathbb{Z}\mathbf{v} + \mathbb{Z}\mathbf{w}$ for some $\|\mathbf{v}\|, \|\mathbf{w}\| < C$.

Now fix such a c_0 and Λ. Let $\mathbf{v}_0 \in \Lambda$ be such that

$$\|\mathbf{v}_0\| = \inf\{\|\mathbf{x}\| \mid \mathbf{x} \in \Lambda \setminus \{\mathbf{0}\}\}.$$

Once $\mathbf{v}_0$ is found, let $\mathbf{w}_0 \in \Lambda$[1] be such that

$$\mathrm{dist}(\mathbf{w}_0, \mathbb{R}\mathbf{v}_0) = \inf\{\mathrm{dist}(\mathbf{x}, \mathbb{R}\mathbf{v}_0) \mid \mathbf{x} \in \Lambda \setminus \mathbb{R}\mathbf{v}_0\}.$$

Note that $\Lambda = \mathbb{Z}\mathbf{v}_0 + \mathbb{Z}\mathbf{w}_0$. Indeed, if $\Lambda = \mathbb{Z}\mathbf{v}_1 + \mathbb{Z}\mathbf{w}_1$ then $\mathbf{v}_0 = a\mathbf{v}_1 + b\mathbf{w}_1$ with $\gcd(a,b) = 1$. Then there exists $\mathbf{w}_0' \in \Lambda$ such that $\Lambda = \mathbb{Z}\mathbf{v}_0 + \mathbb{Z}\mathbf{w}_0'$. Write $\mathbf{w}_0 = c\mathbf{v}_0 + d\mathbf{w}_0'$, then it follows from the definition that $d = \pm 1$. So $\mathbf{w}_0'$ can be written as integral combinations of $\mathbf{v}_0$ and $\mathbf{w}_0$. Thus $\Lambda = \mathbb{Z}\mathbf{v}_0 + \mathbb{Z}\mathbf{w}_0$.

It remains to give an upper bound on $\|\mathbf{v}_0\|$ and $\mathrm{dist}(\mathbf{w}_0, \mathbb{R}\mathbf{v}_0)$ in terms of c_0 (replacing $\mathbf{w}_0$ by $\mathbf{w}_0 - n\mathbf{v}_0$ for suitable n_0 would give an upper bound for $\|\mathbf{w}_0\|$).

[1] As remarked by H.Li, one can simply take $\mathbf{w}_0$ to be any vector such that $\Lambda = \mathbb{Z}\mathbf{v}_0 + \mathbb{Z}\mathbf{w}_0$. As the covolume of Λ is one, the distance from $\mathbf{w}_0$ to $\mathbb{R}\mathbf{v}_0$ must be bounded from above. This gives a shorter proof.

Let $F := [0,2) \times [0,2)$. Then $\mathrm{Leb}(F) > 1$. By Lemma 1.3, for some non-zero $\mathbf{v} \in \Lambda$, $\mathbf{v} + F \cap F \neq \emptyset$. In other words, $\mathbf{v} = \mathbf{x}_1 - \mathbf{x}_2$ for some $\mathbf{x}_i \in F$. Thus

$$\|\mathbf{v}_0\| \leq \|\mathbf{v}\| \leq 2\sqrt{2}.$$

Also note that $\Lambda \cap \mathbb{R}\mathbf{v}_0 = \mathbb{Z}\mathbf{v}_0$ (such $\mathbf{v}_0$ is called primitive).

Pick some unit vector $\mathbf{y}_0$ orthogonal to $\mathbf{v}_0$. Let $F' := (0,1)\mathbf{v}_0 + (0,C)\mathbf{y}_0$ with $C := \frac{2}{c_0}$. Then

$$\mathrm{Leb}(F') = \frac{2\|\mathbf{v}_0\|}{c_0} > 1.$$

Thus we find some non-zero $\mathbf{w} \in (F' - F') \cap \Lambda$. Hence $\mathbf{w} = w_1\mathbf{v}_0 + w_2\mathbf{y}_0$ for some $w_1 \in (-1,1)$ and $w_2 \in (-C,C)$. If $w_2 = 0$, then w_1 has to be integral, so is also 0. This contradicts against the fact that $\mathbf{w}$ is nonzero. So $\mathbf{w} \notin \mathbb{R}\mathbf{v}_0$. Moreover,

$$\mathrm{dist}(\mathbf{w}_0, \mathbb{R}\mathbf{v}_0) \leq \mathrm{dist}(\mathbf{w}, \mathbb{R}\mathbf{v}_0) = |w_2| \leq C = \frac{2}{c_0}.$$

So we are done. □

Corollary 1.8. X_2 *is non-compact.*

A subset B of X_2 is said to be **bounded** iff there exists $c > 0$ such that $\mathrm{sys}(\Lambda) > c$ for every $\Lambda \in B$. Otherwise we say that B is **unbounded**. So a subset is bounded iff it is precompact by Mahler's criterion.

A sequence $(x_n)_{n\in\mathbb{Z}^+}$ (or a subset indexed by positive real numbers $(x_t)_{t\in\mathbb{R}^+}$) in a topological space X is said to be **divergent** iff $\lim_{n\to+\infty} \mathrm{sys}(x_n) = 0$ (resp. $\lim_{t\to+\infty} = 0$). By Mahler's criterion, $(x_n)_{n\in\mathbb{Z}^+}$ is divergent iff for any compact subset $C \subset X$, there exists N such that for every $n > N$ or every $t > N$, $x_n \notin C$.

1.4. **Group action.** The set $\mathbf{SL}_2(\mathbb{R}) := \{$2-by-2 matrices with determinant 1$\}$ is naturally a topological space (subspace topology from $\mathbb{R}^4$) as well as a group (matrix multiplication). It is a **topological group** since

$$\begin{aligned} \mathbf{SL}_2(\mathbb{R}) \times \mathbf{SL}_2(\mathbb{R}) &\to \mathbf{SL}_2(\mathbb{R}) \\ (g,h) &\mapsto gh \end{aligned}$$

and $g \mapsto g^{-1}$ from $\mathbf{SL}_2(\mathbb{R})$ to itself are continuous.

The group $\mathbf{SL}_2(\mathbb{R})$ acts on X_2 by $(g,\Lambda) \mapsto g\Lambda := \{g\mathbf{v},\ \mathbf{v} \in \Lambda\}$. The action is continuous in the sense that

$$\begin{aligned} \mathbf{SL}_2(\mathbb{R}) \times \mathrm{X}_2 &\to \mathrm{X}_2 \\ (g,\Lambda) &\mapsto g\Lambda \end{aligned}$$

is continuous.

Lemma 1.9. *The map* $g \mapsto g.\mathbb{Z}^2$ *from* $\mathbf{SL}_2(\mathbb{R})$ *to* X_2 *is continuous and open. Moreover, it factors through a a homeomorphism* $\mathbf{SL}_2(\mathbb{R})/\mathbf{SL}_2(\mathbb{Z}) \to \mathrm{X}_2$.

There are a few subgroups of $\mathbf{SL}_2(\mathbb{R})$ that we are particularly interested in. First,

$$\mathrm{U}^+ := \left\{ \mathbf{u}_t^+ := \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \middle| \ t \in \mathbb{R} \right\}$$

is a one-parameter subgroup (that is, $t \mapsto \mathbf{u}_t^+$ from $(\mathbb{R},+)$ to U^+ gives an isomorphism of topological groups) consist of unipotent matrices. Its action on X_2 is sometimes referred as a horocycle/unipotent flow. Also,

$$\mathrm{A} := \left\{ \mathbf{a}_t := \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \middle| \ t \in \mathbb{R} \right\}$$

is a one-parameter subgroup consisting of diagonal matrices. Its action on X_2 is sometimes called a geodesic/diagonal flow.

1.4.1. *An explicit metric on* X_2. Once we realize X_2 as a homogeneous space, we can equip it with a metric as follows.

For $A \in \mathbf{SL}_2(\mathbb{R})$, let $\|A\|_{\mathrm{op}}$ denotes the operator norm w.r.t. Euclidean norm:

$$\|A\|_{\mathrm{op}} := \sup_{\mathbf{v} \neq 0 \in \mathbb{R}^2} \frac{\|A.\mathbf{v}\|}{\|\mathbf{v}\|}.$$

Define a metric on $\mathbf{SL}_2(\mathbb{R})$ by

$$\mathrm{dist}(g,h) := \log\left\{1 + \left\|gh^{-1} - \mathrm{I}_2\right\|_{\mathrm{op}} + \left\|hg^{-1} - \mathrm{I}_2\right\|_{\mathrm{op}}\right\}$$

Once can verify that $\mathrm{dist}(g\gamma, h\gamma)$ for every $\gamma \in \mathbf{SL}_2(\mathbb{R})$. Then,

$$\mathrm{dist}(g\mathbb{Z}^2, h\mathbb{Z}^2) := \inf\left\{\mathrm{dist}(g, h\gamma) \mid \gamma \in \mathbf{SL}_2(\mathbb{Z})\right\}$$

define a metric on $\mathbf{SL}_2(\mathbb{R})/\,\mathbf{SL}_2(\mathbb{Z})$.

1.5. **Dani correspondence.** For a real number α, let

$$\Lambda_\alpha := \mathbb{Z}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \mathbb{Z}\begin{bmatrix} \alpha \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}\mathbb{Z}^2 = \mathbf{u}^+_\alpha \mathbb{Z}^2,$$

a unimodular lattice (in X_2).

Lemma 1.10 (Dani correspondence)**.** *A real number* α *is badly approximable iff* $(\mathbf{a}_t\Lambda_\alpha)_{t>0}$ *is bounded in* X_2.

Note that for every α, the full orbit $(\mathbf{a}_t\Lambda_\alpha)_{t\in\mathbb{R}}$ is unbounded. Actually, $(\mathbf{a}_t\Lambda_\alpha)$ as $t \to -\infty$ diverges.

Proof. For every $(x,y)^{\mathrm{tr}} \in \mathbf{a}_t\Lambda_\alpha$, there exists $(m,n) \in \mathbb{Z}^2$ such that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e^t(m+n\alpha) \\ e^{-t}n \end{bmatrix}.$$

So for $\varepsilon > 0$, $\mathrm{sys}(\mathbf{a}_t\Lambda_\alpha) \geq \varepsilon$ iff for every $(m,n) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$,

$$e^t(m+n\alpha) \geq \varepsilon,\ e^{-t}n \geq \varepsilon. \tag{1}$$

Assume α is bad, namely, there exists $c_0 \in (0,1)$ such that for every $(p,q) \in \mathbb{Z}^2$ with $q \neq 0$, $|q|\,|p+q\alpha| > c_0$. So for $\mathbf{0} \neq (x,y)^{\mathrm{tr}} = (e^t(m+n\alpha), e^{-t}n)^{\mathrm{tr}} \in \mathbf{a}_t\Lambda_\alpha$, if $y \neq 0$, then

$$|xy| = |n||m+n\alpha| > c_0,\ \text{implying } (x^2+y^2)^{\frac{1}{2}} \geq \sqrt{2|xy|} > \sqrt{2c_0}.$$

If $y = 0$, then $(x,y)^{\mathrm{tr}} = (e^t m, 0)^{\mathrm{tr}}$. Hence $\|(x,y)^{\mathrm{tr}}\| \geq 1$. Anyway, we have shown that every non-zero vector has norm at least $\sqrt{c_0}$.

Conversely, suppose $\mathrm{sys}(\mathbf{a}_t\Lambda_\alpha) > c_1 > 0$ for all $t > 0$. For every $(p,q) \in \mathbb{Z}^2$ with $q > 0$, take $t_q > 0$ such that $e^{t_q} = \frac{2q}{c_1}$. Then, $\|(e^{t_q}(p+q\alpha), e^{-t_q}q)\| \geq c_1$ by Equa.(1). But

$$|e^{-t}q|^2 \leq \frac{c_1^2}{4}.$$

So

$$|e^{t_q}(p+q\alpha)| \geq \frac{\sqrt{3}}{2}c_1 \implies q\,|p+q\alpha| \geq \frac{\sqrt{3}}{4}c_1^2.$$

The proof is now complete.

□

1.6. **Invariant measures on** X_2**.**

Definition 1.11. *Let* $f : X \to Y$ *be a continuous map between two topological spaces* X *and* Y*. Given a measure* μ *on* $(X, \mathscr{B}_X)$ *(*$\mathscr{B}_X$ *is the Borel* σ*-algebra, the smallest* σ*-algebra containing all open subsets), we define* $f_*\mu$ *to be a measure on* Y *by* $f_*\mu(E) := \mu(f^{-1}(E))$ *for every* $E \in \mathscr{B}_Y$*. If* $X = Y$ *and* $f_*\mu = \mu$*, we say that* f *preserves the measure* μ*. If* G *is a group acting on* X *by homeomorphisms such that* $g_*\mu = \mu$ *for every* $g \in G$*, then we say that* μ *is* G*-invariant.*

Lemma 1.12. *There exists a locally finite* $\mathbf{SL}_2(\mathbb{R})$*-invariant measure* $\mathrm{m}_{\mathrm{X}_2}$ *on* X_2*.*

There are different ways to see the existence of $\mathrm{m}_{\mathrm{X}_2}$. For instance, one may equip $\mathbf{SL}_2(\mathbb{R})$ with a right invariant Riemannian metric and then $\mathbf{SL}_2(\mathbb{R})/\mathbf{SL}_2(\mathbb{Z})$ will inherit a Riemannian metric. One can check that the volume form induced from such a metric is $\mathbf{SL}_2(\mathbb{R})$-invariant. We will give an explicit construction of an invariant measure on $\mathbf{SL}_2(\mathbb{R})$ and then induce one on the quotient space in the next subsection. What is less trivial is that:

Theorem 1.13. $\mathrm{m}_{\mathrm{X}_2}$ *is a finite measure.*

A proof will probably be given in next lecture.

Henceforth, we normalize $\mathrm{m}_{\mathrm{X}_2}$ to be a **probability measure**, namely, $\mathrm{m}_{\mathrm{X}_2}(\mathrm{X}_2) = 1$.

1.7. A construction of the invariant measure.

1.7.1. *Explicit construction of invariant measures on* $\mathbf{SL}_2(\mathbb{R})$. Let

$$\mathcal{O}_1 := \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \mathbf{SL}_2(\mathbb{R}) \;\middle|\; x \neq 0 \right\}, \quad \mathcal{O}_2 := \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \mathbf{SL}_2(\mathbb{R}) \;\middle|\; z \neq 0 \right\}$$

They can be parametrized by domains in Euclidean spaces. The φ_i's below are homeomorphisms:

$$\mathcal{O}_1' := \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; x \neq 0 \right\} \xrightarrow[\simeq]{\varphi_1} \mathcal{O}_1$$

$$\mathcal{O}_2' := \left\{ (x,z,w) \in \mathbb{R}^3 \;\middle|\; z \neq 0 \right\} \xrightarrow[\simeq]{\varphi_2} \mathcal{O}_2$$

where

$$\varphi_1(x,y,z) := \begin{bmatrix} x & y \\ z & \frac{1+yz}{x} \end{bmatrix}, \; \varphi_2(x,z,w) := \begin{bmatrix} x & \frac{xw-1}{z} \\ z & w \end{bmatrix}.$$

Lemma 1.14. *The map* $\varphi_{12}(x,y,z) := (x, z, \frac{1+yz}{x})$ *from* $\{(x,y,z) \in \mathcal{O}_1,\ z \neq 0\}$ *to* $\{(x,z,w) \in \mathcal{O}_2,\ x \neq 0\}$ *sends* $(\varphi_{12})_* \left|\frac{\mathrm{dxdydz}}{x}\right| = \left|\frac{\mathrm{dxdzdw}}{z}\right|$. *Therefore*

$$(\varphi_1)_* \left|\frac{\mathrm{dxdydz}}{x}\right| = (\varphi_2)_* \left|\frac{\mathrm{dxdzdw}}{z}\right|$$

defines a locally finite measure on $\mathbf{SL}_2(\mathbb{R})$. *Also note that* $\{(x,y,z) \in \mathcal{O}_1,\ z = 0\}$ *has measure zero under* $\left|\frac{\mathrm{dxdydz}}{x}\right|$. *Similarly* $\{(x,z,w) \in \mathcal{O}_2,\ x = 0\}$ *has measure zero under* $\left|\frac{\mathrm{dxdzdw}}{z}\right|$.

Proof. Direct calculation. Note that by differentiating $xw - yz = 1$, one obtains $w\mathrm{dx} + x\mathrm{dw} = y\mathrm{dz} + z\mathrm{dy}$. □

Let $\mathrm{m}_{\mathbf{SL}_2(\mathbb{R})}$ denote this measure.

1.7.2. *Invariance property.* Define

$$\mathrm{U}^+ := \left\{ \mathbf{u}_t^+ := \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

and

$$\mathrm{U}^- := \left\{ \mathbf{u}_t^- := \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

Lemma 1.15. $\mathbf{SL}_2(\mathbb{R})$ *is generated by the two subgroups* U^+ *and* U^-.

Proof. Left as exercise. □

By restricting to $\mathcal{O}_1$ or $\mathcal{O}_2$ respectively, it is easy to verify that

Lemma 1.16. $\mathrm{m}_{\mathbf{SL}_2(\mathbb{R})}$ *is invariant under the left multiplication by* $\mathbf{SL}_2(\mathbb{R})$.

By similar reasoning[2], using additionally

$$\mathcal{O}_3 := \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \mathbf{SL}_2(\mathbb{R}) \;\middle|\; y \neq 0 \right\},$$

one can show that

[2] Alternatively, as remarked by H.Li, one can verify the invariance of measure under the transpose map on $\mathcal{O}_1$. Then right invariance then follows from the left invariance. We will make use of this symmetry again later.

Lemma 1.17. $\mathrm{m}_{\mathbf{SL}_2(\mathbb{R})}$ *is also invariant under the right multiplication by* $\mathbf{SL}_2(\mathbb{R})$*.*

1.7.3. *Strict fundamental domain.* A Borel subset $\mathcal{F} \subset \mathbf{SL}_2(\mathbb{R})$ is said to be a **strict fundamental domain** for $\mathbf{SL}_2(\mathbb{Z})$ iff

$$\mathbf{SL}_2(\mathbb{R}) = \bigsqcup_{\gamma \in \mathbf{SL}_2(\mathbb{Z})} \mathcal{F} \cdot \gamma.$$

Lemma 1.18. *Strict fundamental domain exists.*

Proof. First we choose a small open neighborhood $\mathcal{N}$ of identity in $\mathbf{SL}_2(\mathbb{R})$ such that $\mathcal{N}\gamma \cap \mathcal{N} = \emptyset$ for all non-identity element γ in $\mathbf{SL}_2(\mathbb{Z})$. Then choose a sequence $(g_n) \subset \mathbf{SL}_2(\mathbb{R})$ such that

$$\mathbf{SL}_2(\mathbb{R}) = \bigcup g_n \cdot \mathcal{N}.$$

Then we define

$$\begin{aligned} V_1 &:= g_1\mathcal{N} \\ V_2 &:= g_2\mathcal{N} \setminus g_1\mathcal{N}\Gamma \\ V_3 &:= g_3\mathcal{N} \setminus (g_1\mathcal{N}\Gamma \cup g_2\mathcal{N}\Gamma) \\ &.... \end{aligned}$$

From the definition, V_2 is in the complement of $V_1\Gamma$, V_3 is in the complement of $(V_1 \cup V_2)\Gamma$.... Therefore, $V_i \cap V_j\gamma = \emptyset$ for every $i \neq j$ and $\gamma \in \Gamma$. Moreover, by the choice of $\mathcal{N}$, $V_i \cap V_i\gamma = \emptyset$ for non-identity $\gamma \in \mathbf{SL}_2(\mathbb{Z})$. Thus if we let

$$\mathcal{F} := \bigcup_{i=1}^{\infty} V_i,$$

then $\mathcal{F} \cap \mathcal{F}\gamma = \emptyset$ for every $\gamma_{\neq \mathrm{id}} \in \mathbf{SL}_2(\mathbb{Z})$. On the other hand, for $g \in \mathbf{SL}_2(\mathbb{R})$, if n_g is the smallest positive integer n such that $g \in g_n\mathcal{N}\Gamma$, then $g \in V_{n_g}\Gamma \subset \mathcal{F}\Gamma$ by the definition of V_n's. □

1.7.4. *The invariant measure on the quotient.* Fix some strict fundamental domain $\mathcal{F}$, let $\mathrm{m}_{\mathcal{F}}$ be the restriction of $\mathrm{m}_{\mathbf{SL}_2(\mathbb{R})}$ to $\mathcal{F}$. Let $\pi : \mathbf{SL}_2(\mathbb{R}) \to \mathbf{SL}_2(\mathbb{R})/\mathbf{SL}_2(\mathbb{Z})$ be the natural quotient and let $\pi_{\mathcal{F}}$ denote the induced bijection $\mathcal{F} \to \mathbf{SL}_2(\mathbb{R})/\mathbf{SL}_2(\mathbb{Z})$. Let $\mathrm{m}_{[\mathcal{F}]} := (\pi_{\mathcal{F}})_*\mathrm{m}_{\mathcal{F}}$.

Lemma 1.19. *If* $\mathcal{O} \subset \mathbf{SL}_2(\mathbb{R})$ *is such that* π *restricted to* $\mathcal{O}$ *is injective, then*

$$(\pi_{\mathcal{O}})_* \left(\mathrm{m}_{\mathbf{SL}_2(\mathbb{R})}|_{\mathcal{O}}\right) = \mathrm{m}_{[\mathcal{F}]}|_{\pi(\mathcal{O})}.$$

Consequently, $\mathrm{m}_{[\mathcal{F}]}$ *is independent of the choice of strict fundamental domain and* $\mathrm{m}_{[\mathcal{F}]}$ *is invariant under the left action of* $\mathbf{SL}_2(\mathbb{R})$*.*

Proof. It suffices to show $\mathrm{m}_{\mathbf{SL}_2(\mathbb{R})}(\mathcal{O}) = \mathrm{m}_{\mathbf{SL}_2(\mathbb{R})}(\pi_{\mathcal{F}}^{-1}(\pi(\mathcal{O})))$ for every such $\mathcal{O}$ as in the statement.

For every $\gamma \in \mathbf{SL}_2(\mathbb{Z})$, let

$$\mathcal{O}_\gamma := \{x \in \mathcal{O} \mid x\gamma \in \mathcal{F}\}.$$

By assumption, elements from $(\mathcal{O}_\gamma)_{\gamma \in \mathbf{SL}_2(\mathbb{Z})}$ or $(\mathcal{O}_\gamma \cdot \gamma)_{\gamma \in \mathbf{SL}_2(\mathbb{Z})}$ are disjoint from each other. Hence

$$\mathrm{m}_{\mathbf{SL}_2(\mathbb{R})}(\mathcal{O}) = \mathrm{m}_{\mathbf{SL}_2(\mathbb{R})}(\mathcal{O}_\gamma) = \mathrm{m}_{\mathbf{SL}_2(\mathbb{R})}(\mathcal{O}_\gamma \cdot \gamma) = \mathrm{m}_{\mathcal{F}}(\mathcal{O}).$$

□

This finishes the proof of Lemma 1.12. The local finiteness also follows from the lemma above and the fact that $\mathrm{m}_{\mathbf{SL}_2(\mathbb{R})}$ is locally finite.

1.8. **Ergodicity and mixing.**

Definition 1.20. *The action of* $\mathrm{A} \curvearrowright (\mathrm{X}_2, \mathrm{m}_{\mathrm{X}_2})$ *is said to be*

- ***ergodic*** *iff for every Borel subset* $B \subset \mathrm{X}_2$ *that is* A*-invariant (i.e.,* $a.B = B$ *for every* $a \in \mathrm{A}$*), one has* $\mathrm{m}_{\mathrm{X}_2}(B) = 0$ *or* $\mathrm{m}_{\mathrm{X}_2}(\mathrm{X}_2 \setminus B) = 0$*;*
- ***mixing*** *iff for every divergent sequence* $(a_n) \in \mathrm{A}$ *and Borel subsets* B, C*, one has*

$$\lim_{n\to\infty} \mathrm{m}_{\mathrm{X}_2}(B \cap a_n^{-1}.C) = \mathrm{m}_{\mathrm{X}_2}(B)\mathrm{m}_{\mathrm{X}_2}(C).$$

Lemma 1.21. *Mixing implies ergodicity.*

Proof. Indeed, let B be an A-invariant subset and let (a_n) be a divergent sequence in A. Then by mixing,

$$\lim_{n\to\infty} \mathrm{m}_{\mathrm{X}_2}(B \cap a_n^{-1}.B) = \mathrm{m}_{\mathrm{X}_2}(B)^2.$$

By A-invariance, the left hand side is $\mathrm{m}_{\mathrm{X}_2}(B)$. Then $\mathrm{m}_{\mathrm{X}_2}(B)^2 = \mathrm{m}_{\mathrm{X}_2}(B)$ implies $\mathrm{m}_{\mathrm{X}_2}(B) = 0$ or 1. So we are done. □

We are going to prove that the A-action on X_2 is mixing via a little functional analysis.

Theorem 1.22. *The action of* $\mathrm{A} \curvearrowright (\mathrm{X}_2, \mathrm{m}_{\mathrm{X}_2})$ *is mixing.*

1.9. **The associated unitary representation.** Let

$$\mathrm{L}^2(\mathrm{X}_2, \mathrm{m}_{\mathrm{X}_2}) := \left\{ f : \mathrm{X}_2 \to \mathbb{C} \text{ measurable } \middle| \int |f|^2 \mathrm{m}_{\mathrm{X}_2} < +\infty \right\};$$
$$\mathrm{L}^2_0(\mathrm{X}_2, \mathrm{m}_{\mathrm{X}_2}) := \left\{ f \in \mathrm{L}^2(\mathrm{X}_2, \mathrm{m}_{\mathrm{X}_2}) \middle| \int f \mathrm{m}_{\mathrm{X}_2} = 0 \right\}.$$

(note that L^2 functions are in L^1 since $\mathrm{m}_{\mathrm{X}_2}$ is finite) with inner product denoted by

$$\langle f, g \rangle := \int_{\mathrm{X}_2} f(x)\overline{g(x)} \mathrm{m}_{\mathrm{X}_2}(x)$$

where $\bar{b}$ denotes the complex conjugate of a complex number b. Also, $\|f\|_2 := \sqrt{\langle f, f \rangle}$.

As usual, we identify two functions $f, g \in \mathrm{L}^2_0(\mathrm{X}_2, \mathrm{m}_{\mathrm{X}_2})$ if they are equal almost surely. Then $\mathrm{L}^2_0(\mathrm{X}_2, \mathrm{m}_{\mathrm{X}_2})$ with this inner product is a separable (i.e., has a countable dense subset) Hilbert space.

Note that the $\mathbf{SL}_2(\mathbb{R})$ action on $(\mathrm{X}_2, \mathrm{m}_{\mathrm{X}_2})$ induces an action of $\mathbf{SL}_2(\mathbb{R})$ on $\mathrm{L}^2_0(\mathrm{X}_2, \mathrm{m}_{\mathrm{X}_2})$ defined by

$$U_g f(x) := f(g^{-1}x).$$

Lemma 1.23. *The action has the following properties:*

1. *for each* $g \in \mathbf{SL}_2(\mathbb{R})$, $U_g : \mathrm{L}^2_0(\mathrm{X}_2, \mathrm{m}_{\mathrm{X}_2}) \to \mathrm{L}^2_0(\mathrm{X}_2, \mathrm{m}_{\mathrm{X}_2})$ *is a unitary operator;*
2. *for every* $\varepsilon > 0$ *and* $f \in \mathrm{L}^2_0(\mathrm{X}_2, \mathrm{m}_{\mathrm{X}_2})$, *there exists a neighborhood* $\mathcal{O}_\varepsilon$ *of the identity matrix in* $\mathbf{SL}_2(\mathbb{R})$ *such that for every* $g \in \mathcal{O}_\varepsilon$,

$$\|U_g f - f\|_2 \leq \varepsilon.$$

Proof. Take $g \in \mathbf{SL}_2(\mathbb{R})$. Since the action of g preserves $\mathrm{m}_{\mathrm{X}_2}$, we have $\int f(gx)\mathrm{m}_{\mathrm{X}_2}(x) = \int f(x)\mathrm{m}_{\mathrm{X}_2}(x)$ for every integrable function f. For $\phi \in \mathrm{L}^2(\mathrm{X}_2, \mathrm{m}_{\mathrm{X}_2})$, by applying this equality to $f = |\phi|^2$, we see that $\|U_g\phi\|_2 = \|\phi\|_2$.

For the second part, note that the set $C_c(\mathrm{X}_2)$ of compactly supported functions are dense in $\mathrm{L}^2(\mathrm{X}_2, \mathrm{m}_{\mathrm{X}_2})$ (for instance, see Theorem 3.14 in Rudin's book Real and complex analysis).

For every $\varepsilon > 0$ and $f \in \mathrm{L}^2(\mathrm{X}_2, \mathrm{m}_{\mathrm{X}_2})$, find $\phi \in C_c(\mathrm{X}_2)$ such that $\|\phi - f\|_2 \leq 0.1\varepsilon$. Since ϕ is uniformly continuous, find $\delta > 0$ such that $d(x, y) < \delta \implies |\phi(x) - \phi(y)| < 0.1\varepsilon$. Fix a relatively compact neighborhood of identity $\mathcal{O}_0$. Then $\overline{\mathcal{O}_0^{-1}.\operatorname{supp}(\phi)}$ is still compact. Thus, we can find $\mathcal{O}_\varepsilon \subset \mathcal{O}_0$, a neighborhood of the identity, such that for every $g \in \mathcal{O}_\varepsilon$ and $x \in C := \overline{\mathcal{O}_0^{-1}.\operatorname{supp}(\phi)} \cup \operatorname{supp}(\phi)$,

$$|\phi(gx) - \phi(x)| < 0.1\varepsilon.$$

Consequently,

$$\int_{\mathrm{X}_2} |\phi(gx) - \phi(x)|^2 \mathrm{m}_{\mathrm{X}_2}(x) = \int_C |\phi(gx) - \phi(x)|^2 \mathrm{m}_{\mathrm{X}_2}(x) \leq (0.1\varepsilon)^2,$$
$$\text{implying } \|U_g\phi - \phi\|_2 < 0.1\varepsilon.$$

Therefore, for $g \in \mathcal{O}_\varepsilon$

$$\|U_g f - f\|_2 \leq \|U_g\phi - \phi\|_2 + \|U_g\phi - U_g f\| + \|g - f\| \leq 0.1\varepsilon + 0.1\varepsilon + 0.1\varepsilon < \varepsilon.$$

So the proof completes. □

1.10. **Mixing of the geodesic flow.** In this subsection we prove Theorem 1.22. We need to show that for $\phi, \psi \in \mathrm{L}^2_0(\mathrm{X}_2, \mathrm{m}_{\mathrm{X}_2})$ and a divergence sequence $(a_n) \in \mathrm{A}^+$ (namely, assume the $(1,1)$ entries of matrices a_n diverge to $+\infty$. The other case when they diverge to $-\infty$ is similar), one has

$$\lim_{n\to\infty} \langle U_{a_n}\phi, \psi\rangle = 0.$$

For simplicity write $\mathrm{L}^2_0 := \mathrm{L}^2_0(\mathrm{X}_2, \mathrm{m}_{\mathrm{X}_2})$.

1.10.1. *The basics.* As L^2_0 is separable, by applying a diagonal argument, we assume that

$$\lim_{n\to\infty} \langle U_{a_{n_k}}\phi, \psi\rangle \text{ exists, } \forall\, \phi, \psi \in \mathrm{L}^2_0.$$

for some subsequence n_k. It suffices to show that for this subsequence, the limit above is zero for every $\phi, \psi \in \mathrm{L}^2_0$.

For each $\phi \in \mathrm{L}^2_0$, the map $\psi \mapsto \lim_{n\to\infty}\langle U_{a_{n_k}}\phi, \psi\rangle$ is conjugate-linear and bounded since

$$\left|\lim_{n\to\infty} \langle U_{a_{n_k}}\phi, \psi\rangle\right| = \lim_{n\to\infty}\left|\langle U_{a_{n_k}}\phi, \psi\rangle\right| \le \|\phi\|_2 \|\psi\|_2 .$$

By Riesz's lemma, there exists some element in L^2_0, denoted as $E(\phi)$, such that $\langle E(\phi), \psi\rangle = \lim_{n\to\infty}\langle U_{a_{n_k}}\phi, \psi\rangle$ for every $\psi \in \mathrm{L}^2_0$.

Next we note that $\phi \mapsto E(\phi)$ is a bounded operator. Linearity is clear. To show that it is bounded, apply the computation above to $\psi := E(\phi)$,

$$\|E(\phi)\|_2^2 = \langle E(\phi), E(\phi)\rangle \le \|\phi\|_2 \|E(\phi)\|_2 \implies \|E(\phi)\|_2 \le \|\phi\|_2 .$$

Let E^* be the adjoint operator of E, then

$$\langle E^*\phi, \psi\rangle = \lim_{n\to\infty} \langle U_{a_{n_k}^{-1}}\phi, \psi\rangle.$$

1.10.2. *Almost invariant functions are constants.* Next we are going to show that the image of E is fixed by $\mathbf{SL}_2(\mathbb{R})$ and is hence zero by the following lemma

Lemma 1.24. *Let $f \in \mathrm{L}^2_0$. If for every $g \in \mathbf{SL}_2(\mathbb{R})$, $f(g.x) = f(x)$ for almost every $x \in \mathrm{X}_2$, then f is a constant function a.e.*

Proof. Consider the set

$$F := \{(g, x) \in \mathbf{SL}_2(\mathbb{R}) \mid f(g.x) \neq f(x)\}.$$

By Fubini theorem, $\mathrm{m}_{\mathrm{X}_2}(F) = 0$. Let $F_x := \{g \in \mathbf{SL}_2(\mathbb{R}) \mid f(g.x) \neq f(x)\}$. Apply Fubini again

$$\mathrm{m}_{\mathrm{X}_2}(F) = \int_{x\in\mathrm{X}_2} \mathrm{m}_{\mathbf{SL}_2(\mathbb{R})}(F_x)\, \mathrm{m}_{\mathrm{X}_2}(x)$$

So there exists $x_0 \in \mathrm{X}_2$ such that for almost all $g \in \mathbf{SL}_2(\mathbb{R})$, $f(g.x_0) = f(x_0)$. Thus f is equal to $f(x_0)$ a.e. □

Actually, one only needs to show the invariance by the following two special subgroups. Recall

$$\mathrm{U}^+ := \left\{ \mathbf{u}_t^+ := \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \middle| \; t \in \mathbb{R} \right\}.$$

and

$$\mathrm{U}^- := \left\{ \mathbf{u}_t^- := \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \middle| \; t \in \mathbb{R} \right\}.$$

Also recall from Lemma 1.15 that $\mathbf{SL}_2(\mathbb{R})$ is generated by U^+ and U^-.

1.10.3. *The easy part.* For simplicity we assume $n_k = k$.

Elements from U^+ and U^- enjoy the following properties

$$\lim_{n\to\infty} a_n^{-1} u a_n = \mathrm{I}_2,\ \forall\, u \in \mathrm{U}^+, \quad \lim_{n\to\infty} a_n v a_n^{-1} = \mathrm{I}_2,\ \forall\, v \in \mathrm{U}^-.$$

Combined with Lemma 1.23, for an element $u \in \mathrm{U}^+$, one gets

$$\begin{aligned}\langle U_u E(\phi), \psi\rangle &= \lim_{n\to\infty} \langle U_u U_{a_n}\phi, \psi\rangle = \lim_{n\to\infty} \langle U_{a_n^{-1}ua_n}\phi, U_{a_n^{-1}}\psi\rangle \\ &= \lim_{n\to\infty} \langle \phi, U_{a_n^{-1}}\psi\rangle = \langle E(\phi), \psi\rangle.\end{aligned}$$

for all $\phi, \psi \in \mathrm{L}^2_0$. Thus $U_u \circ E = E$.

Similarly, for $v \in \mathrm{U}^-$, one has $U_v \circ E^* = E^*$.

1.10.4. *The trick.*

Lemma 1.25. *Let E, E^* be as above. Then* $\ker(E) = \ker(E^*)$.

Note that in general the kernel of a linear operator is not the same as its adjoint.

Proof.
$$\begin{aligned}\langle E(\phi), E(\phi)\rangle &= \lim_{n\to\infty} \langle U_{a_n}\phi, E(\phi)\rangle \\ &= \lim_{n\to\infty}\lim_{m\to\infty} \langle U_{a_n}\phi, U_{a_m}\phi\rangle \\ &= \lim_{n\to\infty}\lim_{m\to\infty} \langle U_{a_m^{-1}}\phi, U_{a_n^{-1}}\phi\rangle = \langle E^*(\phi), E^*(\phi)\rangle.\end{aligned}$$

□

By result in last subsubsection, $E \circ (U_v - \mathrm{I}_2) = 0$ for every $v \in \mathrm{U}^-$. By the lemma, $E^* \circ (U_v - \mathrm{I}_2) = 0$. Taking the adjoint, we get $U_v \circ E = E$. So we are done.

1.11. **Another proof of Leb(BAD) being zero.** Now we give an alternative proof of the fact that the set of badly approximable numbers has Lebesgue measure zero. We assume $\mathrm{Leb}(\mathbf{BAD}) > 0$ and derive a contradiction.

We fix some $\varepsilon > 0$ and let

$$\begin{aligned}\mathcal{O}_\varepsilon := &\left\{ \begin{bmatrix} e^s & 0 \\ 0 & e^{-s} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} .\mathbb{Z}^2 \,\middle|\, s, t \in (-\varepsilon, \varepsilon),\ r \in (0,1] \right\} \\ = &\,\mathbf{a}_{(-\varepsilon,\varepsilon)} \mathbf{u}^-_{(-\varepsilon,\varepsilon)} \mathbf{u}^+_{(0,1]} .\mathbb{Z}^2.\end{aligned}$$

Let $\mathrm{Obt} : \mathbf{SL}_2(\mathbb{R}) \to \mathrm{X}_2$ defined by $g \mapsto g.\mathbb{Z}^2$.

Lemma 1.26. *There exists a continuous positive function $\varphi : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times (0,1) \to \mathbb{R}$ such that*

$$\mathrm{m}_{\mathrm{X}_2}|_{\mathcal{O}_\varepsilon} = \mathrm{Obt}_* \left(\varphi |\mathrm{dsdtdr}|\right).$$

For $n \in \mathbb{Z}^+$, define

$$\mathbf{BAD}_n := \left\{ r \in \mathbf{BAD},\ \mathrm{sys}(\mathbf{a}_t.\Lambda_r) \geq \frac{1}{n},\ \forall\, t > 0 \right\}$$

By Dani correspondence, $\mathbf{BAD} = \bigcup_{n\in\mathbb{Z}^+} \mathbf{BAD}_n$. Thus $\mathrm{Leb}(\mathbf{BAD}_{n_0}) > 0$ for some $n_0 \in \mathbb{Z}^+$. Let $\mathcal{O}_\varepsilon(\mathbf{BAD}_{n_0})$ be the subset of $\mathcal{O}_\varepsilon$ where $r \in \mathbf{BAD}_{n_0}$. By Lemma 1.26,

$$\mathrm{m}_{\mathrm{X}_2}(\mathcal{O}_\varepsilon(\mathbf{BAD}_{n_0})) > 0.$$

Let

$$\begin{aligned} B_n &:= \overline{\bigcup_{s\geq n} \mathbf{a}_s.\mathcal{O}_\varepsilon(\mathbf{BAD}_{n_0})} \\ B &:= \bigcap_{n\in\mathbb{Z}^+} B_n = \left\{ x = \lim_{n\to\infty} \mathbf{a}_{s_n}.x_n \text{ for some } (s_n) \to +\infty,\ \{x_n\} \subset \mathcal{O}_\varepsilon(\mathbf{BAD}_{n_0}) \right\}. \end{aligned}$$

Since each B_n contains $\mathbf{a}_s.\mathcal{O}_\varepsilon(\mathbf{BAD}_{n_0})$ for some s, we have

$$\mathrm{m}_{\mathrm{X}_2}(B_n) \geq \mathrm{m}_{\mathrm{X}_2}\left(\mathbf{a}_s.\mathcal{O}_\varepsilon(\mathbf{BAD}_{n_0})\right) = \mathrm{m}_{\mathrm{X}_2}(\mathcal{O}_\varepsilon(\mathbf{BAD}_{n_0}))$$

for every n. Hence $\mathrm{m}_{\mathrm{X}_2}(B) \geq \mathrm{m}_{\mathrm{X}_2}(\mathcal{O}_\varepsilon(\mathbf{BAD}_{n_0})) > 0$.

On the other hand, B is A-invariant. By ergodicity, B has full measure in X_2. But B is bounded, as it is contained in (check this!)

$$\mathbf{a}_{(-\varepsilon,\varepsilon)}.\left\{ \Lambda \in \mathrm{X}_2,\ \mathrm{sys}(\Lambda) \geq \frac{1}{n_0} \right\}.$$

This is a contradiction.

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