LECTURE 2.5, INJECTIVITY RADIUS AND A THICKENING TRICK

RUNLIN ZHANG†

Contents

Notation 1
1. Appendix to lecture 2, injectivity radius. 1
1.1. Injectivity radius 1
1.2. Integration in local coordinates 2
1.3. Uniform mixing in a weak sense 3
1.4. Preparations 3
1.5. Proof of Theorem 1.5 3
References 4

Notation

Let I_2 be the two-by-two identity matrix.

Fix a right invariant metric $d^{\mathbf{SL}_2(\mathbb{R})}$ on $\mathbf{SL}_2(\mathbb{R})$ compatible with its topology. Let d^{X_2} be the induced metric on $\mathbf{SL}_2(\mathbb{R})/\mathbf{SL}_2(\mathbb{Z}) \cong \mathrm{X}_2$ defined by

$$d^{\mathrm{X}_2}(g_1\,\mathbf{SL}_2(\mathbb{Z}), g_2\,\mathbf{SL}_2(\mathbb{Z})) := \inf_{\gamma_1,\gamma_2 \in \mathbf{SL}_2(\mathbb{Z})} d^{\mathbf{SL}_2(\mathbb{R})}(g_1\gamma_1, g_2\gamma_2) = \inf_{\gamma \in \mathbf{SL}_2(\mathbb{Z})} d^{\mathbf{SL}_2(\mathbb{R})}(g_1, g_2\gamma).$$

Note that this inf can actually be obtained for some $\gamma \in \mathbf{SL}_2(\mathbb{Z})$.

For $\delta > 0$ and $x \in \mathrm{X}_2$, let

$$B(\delta) := \left\{ g \in \mathbf{SL}_2(\mathbb{R}) \;\middle|\; d^{\mathbf{SL}_2(\mathbb{R})}(g, \mathrm{I}_2) < \delta \right\}, \quad B_x^{\mathrm{X}_2}(\delta) := \left\{ y \in \mathrm{X}_2 \;\middle|\; d^{\mathrm{X}_2}(x, y) < \delta \right\}.$$

For simplicity, we will write $d := d^{\mathbf{SL}_2(\mathbb{R})}$ and $B_x(\delta) = B_x^{\mathrm{X}_2}(\delta)$. Hopefully no confusion shall arise.

1. Appendix to lecture 2, injectivity radius.

1.1. Injectivity radius.

Definition 1.1. *For $x \in \mathrm{X}_2 \cong \mathbf{SL}_2(\mathbb{R})/\mathbf{SL}_2(\mathbb{Z})$, choose $g_x \in \mathbf{SL}_2(\mathbb{R})$ such that $x = g_x\,\mathbf{SL}_2(\mathbb{Z})$, define*[1]

$$\mathrm{InjRad}(x) := \frac{1}{10} \inf \left\{ d(g_x\gamma, g_x) \mid \gamma_{\neq \mathrm{I}_2} \in \mathbf{SL}_2(\mathbb{Z}) \right\},$$

which is independent of the choice of g_x.

Lemma 1.2. *Let $\mathscr{C} \subset \mathrm{X}_2$ be a compact subset, then there exists $c > 0$ such that $\mathrm{InjRad}(x) > c$ for every $x \in \mathscr{C}$.*

Proof. Since every compact subset of X_2 is contained in the image under $\mathbf{SL}_2(\mathbb{R}) \to \mathbf{SL}_2(\mathbb{R})/\mathbf{SL}_2(\mathbb{Z})$ of some compact subset of X_2, it suffices to show that

$$\inf \left\{ d(\mathrm{I}_2, g\gamma g^{-1}) \;\middle|\; g \in \mathscr{C}',\ \gamma_{\neq \mathrm{I}_2} \in \mathbf{SL}_2(\mathbb{Z}) \right\} > 0$$

for every compact subset $\mathscr{C}'$ of X_2.

Let $\Gamma(1)$ be the collection of $\gamma \in \Gamma \setminus \{\mathrm{I}_2\}$ such that $g\gamma g^{-1} \in \overline{B_1}$ for some $g \in \mathscr{C}'$. Since the map $(g,h) \mapsto g^{-1}hg$ is continuous, we know that the union of $g^{-1}\overline{B(1)}g$ for $g \in \mathscr{C}'$ is compact. Hence $\Gamma(1)$ is a compact subset of a discrete subset $\Gamma \setminus \{\mathrm{I}_2\}$, which must be finite. Say, $\Gamma(1) = \{\gamma_1, ..., \gamma_l\}$.

Then

$$\inf \left\{ d(\mathrm{I}_2, g\gamma g^{-1}) \;\middle|\; g \in \mathscr{C}',\ \gamma_{\neq \mathrm{I}_2} \in \mathbf{SL}_2(\mathbb{Z}) \right\}$$

† Email: zhangrunlinmath@outlook.com.

[1] We are content with this rather coarse definition of injectivity radius here, which might be different from the one you are used to.

is either at least one or is equal to

$$\inf \left\{ d(\mathrm{I}_2, g) \,\middle|\, g \in \bigcup_{g \in \mathscr{C}'} \bigcup_{i=1}^{l} g\gamma_i g^{-1} \right\} > 0.$$

But

$$\bigcup_{g \in \mathscr{C}'} \bigcup_{i=1}^{l} g\gamma_i g^{-1} = \bigcup_{i=1}^{l} \bigcup_{g \in \mathscr{C}'} g\gamma_i g^{-1}$$

is a finite union of compact subsets, and is thus compact. It also does not contain I_2. Therefore it must have positive distance away from I_2. So we are done. □

Lemma 1.3. *For $x \in \mathrm{X}_2$ and $\delta < \mathrm{InjRad}(x)$, the natural map*

$$\begin{aligned} \mathrm{Obt}_x : B(\delta) &\to B(\delta).x \\ g &\mapsto g.x \end{aligned}$$

is an isometry between $(B(\delta), d) \cong (B_x^{\mathrm{X}_2}(\delta), d^{\mathrm{X}_2})$. In particular, $B_x^{\mathrm{X}_2}(\delta) = B(\delta).x$.

Proof. For $g_1, g_2 \in B(\delta)$, we need to show that

$$\inf_{\gamma \in \mathbf{SL}_2(\mathbb{Z})} d(g_1 g_x, g_2 g_x \gamma) = d(g_1, g_2).$$

In different words,

$$d(g_1 g_x, g_2 g_x \gamma) > d(g_1, g_2), \quad \forall\, \gamma_{\neq \mathrm{I}_2} \in \mathbf{SL}_2(\mathbb{Z}).$$

This can be seen from the following inequalities:

$$\begin{aligned} d(g_2 g_x, g_2 g_x \gamma) &> d(g_x, g_x\gamma) - d(g_x, g_2 g_x) - d(g_2 g_x \gamma, g_x \gamma) \\ &= d(g_x, g_x \gamma) - d(\mathrm{I}_2, g_2) - d(g_2, \mathrm{I}_2) \\ &> 10\delta - \delta - \delta = 8\delta. \end{aligned}$$

Then

$$\begin{aligned} d(g_1 g_x, g_2 g_x \gamma) &\geq d(g_2 g_x, g_2 g_x \gamma) - d(g_1 g_x, g_2 g_x) \\ &> d(g_2 g_x, g_2 g_x \gamma) - 2\delta > 8\delta - 2\delta = 6\delta. \end{aligned}$$

But $d(g_1, g_2) < 2\delta$. So we are done. The last claim follows from the definition of the distance function on the quotient. □

1.2. **Integration in local coordinates.** For $\eta > 0$, define

$$\mathcal{O}_\eta := \left\{ \mathbf{a}_r \mathbf{u}_s^- \mathbf{u}_t^+ \,\middle|\, r, s, t \in (-\eta, \eta) \right\}.$$

By explicit calculation, one can show that $\mathcal{O}_\eta$ is an open neighborhood of the identity element in $\mathbf{SL}_2(\mathbb{R})$ for every $\eta > 0$.

We fix $\eta_0 > 0$ small enough such that

$$\begin{aligned} (-\eta_0, \eta_0)^3 &\mapsto \mathcal{O}_{\eta_0} \\ (r, s, t) &\mapsto \mathbf{a}_r \mathbf{u}_s^- \mathbf{u}_t^+ \end{aligned}$$

is a homeomorphism. We find ϕ_{η_0}, a positive continuous function on $[-\eta_0, \eta_0]^3$, such that for every $f \in L^1(\mathbf{SL}_2(\mathbb{R}), \mathrm{m}_{\mathbf{SL}_2(\mathbb{R})})$,

$$\int_{z \in \mathcal{O}_{\eta_0}} f(z) \mathrm{m}_{\mathbf{SL}_2(\mathbb{R})}(z) = \int_{-\eta_0}^{\eta_0} \int_{-\eta_0}^{\eta_0} \int_{-\eta_0}^{\eta_0} f(\mathbf{a}_r \mathbf{u}_s^- \mathbf{u}_t^+)\, \phi_{\eta_0}(r, s, t)\, \mathrm{drdsdt}.$$

Fix a constant $C_1 > 1$ such that $\|\phi_{\eta_0}\|_{\sup} \leq C_1$.

By the relation between $\mathrm{m}_{\mathbf{SL}_2(\mathbb{R})}$ and $\mathrm{m}_{\mathrm{X}_2}$, one can show that

Lemma 1.4. *Let $x \in \mathrm{X}_2$ and $\delta < \mathrm{InjRad}(x)$. Let $0 < \eta < \eta_0$ be such that $\mathcal{O}_\eta \subset B(\delta)$. Then for every $f \in L^1(\mathrm{X}_2, \mathrm{m}_{\mathrm{X}_2})$,*

$$\int_{z \in \mathcal{O}_\eta . x} f(z) \mathrm{m}_{\mathbf{SL}_2(\mathbb{R})}(z) = \int_{-\eta}^{\eta} \int_{-\eta}^{\eta} \int_{-\eta}^{\eta} f(\mathbf{a}_r \mathbf{u}_s^- \mathbf{u}_t^+ . x)\, \phi_{\eta_0}(r, s, t)\, \mathrm{drdsdt}.$$

Proof. This follows from Lemma 1.3. □

1.3. **Uniform mixing in a weak sense.** The main result of this appendix is the following very weak form of equidistribution of expanding unipotent trajectories. The point is the uniformity as the base points vary in a compact subset.

Theorem 1.5. *Fix $y_0 \in \mathrm{X}_2$, $\varepsilon_0 \in (0,1)$ and a compact subset $\mathscr{C}$ of X_2. There exist $\delta, T > 0$ and $M \in 2\mathbb{Z}^+$ such that for every $x \in \mathscr{C}$ and $T' > T$,*

$$\int_{-0.5}^{0.5} \mathbf{1}_{B_{y_0}^{d_{\mathrm{X}_2}}(\varepsilon_0)}(\mathbf{a}_{T'}\mathbf{u}_t^+.x)\mathrm{dt} > \delta$$

Note

$$\int_{-0.5}^{0.5} \mathbf{1}_{B_{y_0}^{d_{\mathrm{X}_2}}(\varepsilon_0)}(\mathbf{a}_{T'}\mathbf{u}_t^+.x)\mathrm{dt} = \mathrm{Leb}\left\{ t \in [-0.5, 0.5] \;\middle|\; \mathbf{a}_{T'}\mathbf{u}_t^+.x \in B_{y_0}^{d_{\mathrm{X}_2}}(\varepsilon_0) \right\}$$

1.4. **Preparations.** Firstly, by Lemma 1.2, we find $0 < \eta_1 < \eta_0$ such that $\mathcal{O}_{\eta_1} \subset B(\delta_0)$ for some $\delta_0 > 0$ that is smaller than $\mathrm{InjRad}(x)$ for all $x \in \mathscr{C}$. Thus, Lemma 1.4 is applicable to every $x \in \mathscr{C}$ and $\eta = \eta_1$.

Then we choose $0 < \eta_2 < \min\{\eta_1, 0.1\}$ such that $\mathcal{O}_{\eta_2} \subset B(0.5\varepsilon_0)$. This has the effect that

Lemma 1.6. *For every $x \in \mathrm{X}_2$, $T \geq 0$ and $r, s \in (-\eta_2, \eta_2)$, one has the following implication:*

$$\mathbf{a}_T.(\mathbf{a}_r\mathbf{u}_s^-.x) \in B_{y_0}(0.5\varepsilon_0) \implies \mathbf{a}_T.x \in B_{y_0}(\varepsilon_0).$$

Proof. Indeed, given $\mathbf{a}_T.(\mathbf{a}_r\mathbf{u}_s^-.x) \in B_{y_0}(0.5\varepsilon_0)$, we have

$$\begin{aligned} d^{\mathrm{X}_2}(\mathbf{a}_T.x, y_0) &\leq d^{\mathrm{X}_2}(\mathbf{a}_T.x, \mathbf{a}_T.\mathbf{a}_r\mathbf{u}_s^-.x) + d^{\mathrm{X}_2}(\mathbf{a}_T.\mathbf{a}_r\mathbf{u}_s^-.x, y_0) \\ &< d^{\mathrm{X}_2}(\mathbf{a}_T.x, \mathbf{a}_r\mathbf{u}_{e^{-2T}s}^-\mathbf{a}_T.x) + 0.5\varepsilon_0 \\ &\leq d(\mathrm{I}_2, \mathbf{a}_r\mathbf{u}_{e^{-2T}s}^-) + 0.5\varepsilon_0 \\ (\because \mathcal{O}_{\eta_2} \subset B(0.5\varepsilon_0)\ ) &< 0.5\varepsilon_0 + 0.5\varepsilon_0 = \varepsilon_0. \end{aligned}$$

□

Next we choose $0 < \eta_3 < \eta_2$ satisfying the following:

Lemma 1.7. *There exists $0 < \eta < \eta_2$ such that for every $x, y \in \mathscr{C}$, the following implication holds:*

$$x \in \mathcal{O}_\eta.y \implies \mathcal{O}_\eta.y \subset \mathcal{O}_{\eta_2}.x.$$

Proof. Choose $0 < \theta < \delta_0$ (the uniform injectivity radius) such that $B(\theta) \subset \mathcal{O}_{\eta_2}$. Then choose $0 < \eta < \eta_2$ such that $\mathcal{O}_\eta \subset B(0.5\theta)$. So

$$x \in \mathcal{O}_\eta.y \implies x \in B(0.5\theta).y \implies y \in B(\theta).x \subset \mathcal{O}_{\eta_2}.x.$$

This completes the proof. □

1.5. **Proof of Theorem 1.5.** Find $M \in 2\mathbb{Z}^+$ large such that $\eta_2^{-1} - 2 \leq M \leq \eta_2^{-1}$. By compactness, find finitely many $\{x_1, ..., x_l\} \subset \mathscr{C}$ such that

$$\mathscr{C} \subset \bigcup_{i=1}^{l} \mathcal{O}_{\eta_3}.x_i.$$

By mixing (Theorem 1.22 from Lecture 2), for each $i = 1, .., l$, we find $T_i > 0$ such that for every $T > T_i$,

$$\begin{aligned} \mathrm{m}_{\mathrm{X}_2}\left(\mathcal{O}_{\eta_3}.x_i \cap \mathbf{a}_T^{-1}B_{y_0}(0.5\varepsilon_0)\right) &> 0.5\mathrm{m}_{\mathrm{X}_2}\left(\mathcal{O}_{\eta_3}.x_i\right)\mathrm{m}_{\mathrm{X}_2}\left(B_{y_0}(0.5\varepsilon_0)\right) \\ &= 0.5\mathrm{m}_{\mathbf{SL}_2(\mathbb{R})}\left(\mathcal{O}_{\eta_3}\right)\mathrm{m}_{\mathrm{X}_2}\left(B_{y_0}(0.5\varepsilon_0)\right). \end{aligned}$$

Let $T := \max\{T_i\}$ and c_1 denote the right hand side. Also, let

$$\delta := \frac{c_1}{C_1 4(\eta_2)^2}.$$

Now take $x \in \mathscr{C}$ and $T' > T$ and let us prove the conclusion.

Find i such that $x \in \mathcal{O}_{\eta_3}.x_i$. By Lemma 1.7, we have $\mathcal{O}_{\eta_3}.x_i \subset \mathcal{O}_{\eta_2}.x$.

So

$$
\begin{aligned}
c_1 &< \mathrm{m}_{\mathrm{X}_2}\left(\mathcal{O}_{\eta_3}.x_i \cap \mathbf{a}_{T'}^{-1} B_{y_0}(0.5\varepsilon_0)\right)\\
&< \mathrm{m}_{\mathrm{X}_2}\left(\mathcal{O}_{\eta_2}.x \cap \mathbf{a}_{T'}^{-1} B_{y_0}(0.5\varepsilon_0)\right)\\
&= \int_{\mathcal{O}_{\eta_2}.x} \mathbf{1}_{B_{y_0}(0.5\varepsilon_0)}(\mathbf{a}_{T'} z)\,\mathrm{m}_{\mathrm{X}_2}(z)\\
\text{(local integration lemma 1.4)} &= \int_{-\eta_2}^{\eta_2}\int_{-\eta_2}^{\eta_2}\int_{-\eta_2}^{\eta_2} \mathbf{1}_{B_{y_0}(0.5\varepsilon_0)}(\mathbf{a}_{T'}\mathbf{a}_r\mathbf{u}_s^{-}\mathbf{u}_t^{+}.x)\phi_{\eta_0}(r,s,t)\mathrm{drdsdt}\\
\text{(boundedness of density function)} &\le C_1\int_{-\eta_2}^{\eta_2}\int_{-\eta_2}^{\eta_2}\int_{-\eta_2}^{\eta_2} \mathbf{1}_{B_{y_0}(0.5\varepsilon_0)}(\mathbf{a}_{T'}\mathbf{a}_r\mathbf{u}_s^{-}\mathbf{u}_t^{+}.x)\mathrm{drdsdt}\\
\text{(Lemma 1.6)} &\le C_1\int_{-\eta_2}^{\eta_2}\int_{-\eta_2}^{\eta_2}\int_{-\eta_2}^{\eta_2} \mathbf{1}_{B_{y_0}(\varepsilon_0)}(\mathbf{a}_{T'}\mathbf{u}_t^{+}.x)\mathrm{drdsdt}\\
&= C_1 4\eta_2^2\int_{-\eta_2}^{\eta_2} \mathbf{1}_{B_{y_0}(\varepsilon_0)}(\mathbf{a}_{T'}\mathbf{u}_t^{+}.x)\mathrm{dt}\\
&< C_1 4\eta_2^2\int_{-0.5}^{0.5} \mathbf{1}_{B_{y_0}(\varepsilon_0)}(\mathbf{a}_{T'}\mathbf{u}_t^{+}.x)\mathrm{dt}
\end{aligned}
$$

Finally we have

$$
\int_{-0.5}^{0.5} \mathbf{1}_{B_{y_0}(\varepsilon_0)}(\mathbf{a}_{T'}\mathbf{u}_t^{+}.x)\mathrm{dt} > \frac{c_1}{C_1 4(\eta_2)^2} = \delta.
$$

References