习题二

April 28, 2024

选取五道题解答。截止日期：5 月 17 日课前。中英文皆可。你们可以互相讨论（当然，我希望你们互相讨论！），或者查阅资料。但是写在纸上/latex 这一过程请务必独立完成。

From Exercises A to J, let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function with $f(\mathbf{0}) = 0$. You are allowed to take $n = 2$ (or $n = 3$) when working on these exercises since we did not define X_n for $n \geq 4$ in the class. Define

$$H_f := \{h \in \mathbf{SL}_n(\mathbb{R}) \mid f(h.\mathbf{v}) = f(\mathbf{v}),\ \forall\, \mathbf{v} \in \mathbb{R}^n\}.$$

Also,

$$m^*(f) := \inf\{|f(\mathbf{v})| \mid \mathbf{v} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}\}$$
$$m^{**}(f) := \inf\{|f(\mathbf{v})| \mid \mathbf{v} \in \mathbb{Z}^n,\ f(\mathbf{v}) \neq 0\}.$$

Exercise A. *If $H_f.\mathbb{Z}^n$ is unbounded in X_n, then $m^*(f) = 0$.*

Exercise B. *If $f : \mathbb{R}^n \to \mathbb{R}$ is surjective and $H_f.\mathbb{Z}^n$ is dense in X_n, then $f(\mathbb{Z}^n)$ is dense in $\mathbb{R}$.*

For

$$\sigma \in S_3 := \{\text{bijections from } \{1,2,3\} \text{ to } \{1,2,3\}\},$$

let $M(\sigma) \in \mathbf{SL}_3(\mathbb{R})$ be defined by specifying the elements on the i-th row and j-th column:

$$M(\sigma)_{ij} := \begin{cases} \operatorname{sgn}(\sigma) & \text{if } \sigma(j) = i; \\ 0 & \text{otherwise} \end{cases}$$

where $\operatorname{sgn}(\sigma) \in \{1, -1\}$ denotes the usual signature of a permutation as you learned in linear algebra. Alternatively, $M(\sigma)\mathbf{e}_i = \operatorname{sgn}(\sigma)\mathbf{e}_{\sigma(i)}$ for $i = 1, 2, 3$ where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ denotes the standard basis of $\mathbb{R}^3$. Note that

$$W_3 := \{M(\sigma) \mid \sigma \in S_3\}$$

is a subgroup of $\mathbf{SL}_3(\mathbb{R})$ normalizing

$$D := \left\{ \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix} \middle| \, t_1, t_2, t_3 \in \mathbb{R},\ t_1 \cdot t_2 \cdot t_3 = 1 \right\}.$$

Thus $W_3 \cdot D = D \cdot W_3$ is a subgroup of $\mathbf{SL}_3(\mathbb{R})$. Let $\mathrm{Nm} : \mathbb{R}^3 \to \mathbb{R}$ be defined by $\mathrm{Nm}(x, y, z) := xyz$.

Exercise C. *Show that $H_{\mathrm{Nm}} = W_3 \cdot D$.*

Exercise D. *Show that $H_{\mathrm{Nm}}.\mathbb{Z}^3$ is unbounded in X_3, yet $m^{**}(\mathrm{Nm}) \neq 0$.*

We say that

(S1) f has property (S1) iff for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $\mathbf{v} \in \mathbb{R}^n$ with $0 < |f(\mathbf{v})| < \delta$, there exists $h \in H_f$ such that $\|h.x\| < \varepsilon$;

(S2) f has property (S2) iff for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $\mathbf{v} \in \mathbb{R}^n$ with $0 \le |f(\mathbf{v})| < \delta$, there exists $h \in H_f$ such that $\|h.x\| < \varepsilon$.

Exercise E. *Assume f has property* (S2)*, then $m^*(f) = 0$ implies that $H_f.\mathbb{Z}^n$ is unbounded in X_n.*

Exercise F. *Assume f has property* (S1)*, then $m^{**}(f) = 0$ implies that $H_f.\mathbb{Z}^n$ is unbounded in X_n.*

Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\pi(x, y) := x$.

Exercise G. *Show that*

$$H_\pi = \left\{ \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix} \,\middle|\, r \in \mathbb{R} \right\}.$$

Exercise H. *The function π defined above has property* (S1) *but not* (S2)*. Also, $m^*(\pi) = 0$ but $H_\pi.\mathbb{Z}^2$ is bounded in X_2.*

Let $Q_0 : \mathbb{R}^3 \to \mathbb{R}$ defined by $Q_0(x, y, z) := x^2 + y^2 - z^2$.

Exercise I. *The function Q_0 satisfies property* (S2)*.*

Exercise J. *Let $f(x, y) := (x - \sqrt{2}y)(x + (\sqrt{2} + 1)y)$. Show that $m^*(f) \neq 0$*[1]*.*

Let d denote your favorite metric on X_2 compatible with the topology. For instance, you can take the one defined in the class.

Exercise K. *Let $\mathbf{a}_t := \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}$. Assume $x \in \mathrm{X}_2$ is such that*

$$\{\mathbf{a}_t.x \mid t \ge 0\} \subset \mathrm{X}_2$$

is bounded. Take another $y \in \mathrm{X}_2$. Show that the following two are equivalent

(1) $\lim_{t \to +\infty} d(\mathbf{a}_t.x, \mathbf{a}_t.y)$ *exists and is* 0*;*

(2) $y = \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}.x$ *for some $r \in \mathbb{R}$.*

Exercise L. *Find an irrational number α and $\varepsilon_0 > 0$ such that $\langle 2^n \alpha \rangle > \varepsilon_0$ for all $n \in \mathbb{Z}_{\ge 0}$.*

On the other hand, Furstenberg's theorem asserts that if α is an irrational number, then $\{2^n 3^m \alpha,\ n, m \in \mathbb{Z}_{\ge 0}\}$ is dense modulo $\mathbb{Z}$. The following few exercises is to walk you through a proof of this fact[2].

Order the set $\{2^n 3^m,\ n, m \in \mathbb{Z}_{\ge 0}\}$ as

$$\{2^n 3^m,\ n, m \in \mathbb{Z}_{\ge 0}\} = \{a_1 < a_2 < a_3 < ...\}$$

Exercise M. *Let α be an irrational number. Show that for every $\varepsilon > 0$, there exists $n, m \in \mathbb{Z}^+$ such that*

$$0 < n\alpha - m < \varepsilon.$$

Exercise N. *Show that* $\lim_n \frac{a_{n+1}}{a_n} = 1$.

Hint: apply the proceeding exercise to $\log(2)/\log(3)$ and $\log(3)/\log(2)$.

For an integer p, let $T_p : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be defined by $x + \mathbb{Z} \mapsto px + \mathbb{Z}$. For an irrational number α, let C_α be the closure of the set

$$\{T_2^n T_3^m(\alpha + \mathbb{Z}) \mid n, m \in \mathbb{Z}_{\ge 0}\} = \{2^n 3^m \alpha + \mathbb{Z} \mid n, m \in \mathbb{Z}_{\ge 0}\} \subset \mathbb{R}/\mathbb{Z}.$$

Exercise O. *Take α, an irrational number. Show that there exists $(\varepsilon_n) \subset C_\alpha - C_\alpha$, $\varepsilon_n \neq 0 + \mathbb{Z}$ such that $\lim_n \varepsilon_n = 0 + \mathbb{Z}$.*

[1] This shows that Oppenheim conjecture fails in two variables.

[2] if you manage to give a proof without doing any of the following exercises, you would still earn the full credit.

Exercise P. *Take α, an irrational number. Show that $C_\alpha - C_\alpha = \mathbb{R}/\mathbb{Z}$. Namely, for every $x+\mathbb{Z} \in \mathbb{R}/\mathbb{Z}$, there exists $a+\mathbb{Z}, b+\mathbb{Z} \in C_\alpha$ such that $a-b \equiv x \pmod{\mathbb{Z}}$.*

Hint: Use the Exercise O and N.
For $l \in \mathbb{Z}^+$ that is coprime to $2,3$, define

$$\Sigma_l := \{2^n 3^m \mid n,m \in \mathbb{Z}_{\geq 0},\ 2^n 3^m \equiv 1 \pmod{l}\}.$$

And order Σ_l as

$$\Sigma_l = \{a_1^{(l)} < a_2^{(l)} < a_3^{(l)} < ...\}$$

Exercise Q. *For $l \in \mathbb{Z}^+$ that is coprime to $2,3$, $\lim_n \frac{a_{n+1}^{(l)}}{a_n^{(l)}} = 1$.*

Exercise R. *Take an irrational number α. If C_α contains a rational number modulo $\mathbb{Z}$, then $C_\alpha = \mathbb{R}/\mathbb{Z}$.*

Hint: similar to the proof of Exercise P. Maybe you also need Exercise Q in the place of Exercise N.

Exercise S. *Let α be an irrational number. Show that C_α must contain a rational point modulo $\mathbb{Z}$.*

Hint: If C_α does not contain any rational number, then for every $l \in \mathbb{Z}^+$, show that[3]

$$C_\alpha \cap \left(C_\alpha - \frac{1}{5^l}\right) \cap \left(C_\alpha - \frac{2}{5^l}\right) \cap .... \cap \left(C_\alpha - \frac{5^l-1}{5^l}\right) \neq \emptyset. \tag{1}$$

How to show this? Certainly, $C_\alpha \cap \left(C_\alpha - \frac{1}{5^l}\right)$ is non-empty by Exercise P. Call $D_1 := C_\alpha \cap (C_\alpha - \frac{1}{5^l})$. Note that

$$C_\alpha \cap (C_\alpha - \frac{1}{5^l}) \cap (C_\alpha - \frac{2}{5^l}) = D_1 \cap \left(D_1 - \frac{1}{5^l}\right).$$

On the other hand D_1 is invariant under $\{T_n,\ n \in \Sigma_{5^l}\}$. If it does not contain any rational point, then you can show that, just as Exercise P, $D_1 - D_1 = \mathbb{R}/\mathbb{Z}$. In particular $D_1 \cap \left(D_1 - \frac{1}{5^l}\right)$ is non-empty. Equa.(1) can be verified by repeating this process.

[3]Here "$-\frac{1}{5^l}$" should be understood as modulo $\mathbb{Z}$.