习题 1

March 23, 2024

选取五道题解答。截止日期：4 月 5 日课前（如果这一周不上课就延迟到 4 月 12 日课前）。中英文皆可。你们可以互相讨论（当然，我希望你们互相讨论！），或者查阅资料。但是写在纸上/latex 这一过程请务必独立完成。

当我引用课程讲义的时候，定理的编号等是按照 https://runlinzhang.github.io/teaching2024sp 上的版本来。

Exercise A. *Prove that there exists $c_0 > 0$ such that for every $q \in \mathbb{Z}^+$, $q^2\langle q\sqrt[3]{2}\rangle > c_0$.*

Exercise B. *Prove that for $n \in \mathbb{Z}^+$, the map $T_n : [0,1) \to [0,1)$ defined by $T_n(x) - nx \in \mathbb{Z}$ preserves the Lebesgue measure. Namely, for every Borel measurable subset E (not just intervals) of $[0,1)$, show that $\mathrm{Leb}(T_n^{-1}(E)) = \mathrm{Leb}(E)$.*

Exercise C. *Prove Cassels' zero-one law (Theorem 1.19 of "Lecture 1") without assuming ψ is non-increasing.*

Exercise D. *Find two non-increasing sequences of positive numbers $(a_n)_{n\in\mathbb{Z}^+}$ and $(b_n)_{n\in\mathbb{Z}^+}$ such that $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n = +\infty$ but $\sum_{n=1}^{\infty} \min\{a_n, b_n\} < +\infty$.*

Exercise E. *Prove Lemma 1.25 from "Lecture 1" using Lemma 1.24 from there.*

Define (N is a positive integer)

$$\begin{aligned} \mathcal{L} &:= \left\{(x,y) \in \mathbb{Z}^2 \,\middle|\, \gcd(x,y) = 1,\ 0 < x < y\right\}, \\ \mathcal{L}_N &:= \{(x,y) \in \mathcal{L} \mid y < N\}. \end{aligned}$$

For every $(x,y) \in \mathcal{L}$, define $\pi(x,y) := (1, \frac{y}{x}) \in \{1\} \times (0,1)$. For every $N \in \mathbb{Z}^+$, define a measure μ_N on $\{1\} \times (0,1)$ by

$$\mu_N := \frac{1}{\#\mathcal{L}_N} \sum_{(x,y)\in\mathcal{L}_N} \delta_{\pi(x,y)}$$

where $\delta_{(x,y)}$ is the Dirac measure supported on (x,y) defined by

$$\delta_{(x,y)}(E) = \begin{cases} 1 & \text{if } (x,y) \in E \\ 0 & \text{if } (x,y) \notin E. \end{cases}$$

Exercise F. *Prove that (μ_N) converges, in the weak$*$ topology, to the standard Lebesgue measure on $\{1\} \times (0,1)$. For simplicity, you are only required to show the following: for every interval $(a,b) \subset (0,1)$, one has*

$$\lim_{N\to\infty} \mu_N(\{1\} \times (a,b)) = b - a.$$

(Hint: adapt the proof of Lemma 1.29 from "Lecture 1".)

Definition 0.1. *Two lattices $\Lambda_1, \Lambda_2 \subset \mathbb{R}^2$ are said to be* ***commensurable*** *iff $\Lambda_1 \cap \Lambda_2$ is a finite-index subgroup in both Λ_1 and Λ_2.*

Exercise G. *Let $\Lambda_0 \in \mathrm{X}_2$ be a unimodular lattice, then the set*

$$\{\Lambda \in \mathrm{X}_2 \mid \Lambda \textit{ is commensurable with } \Lambda_0\}$$

is dense in X_2.

Recall $\mathrm{A} = \left\{ \mathbf{a}_t := \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\}$.

Exercise H. *Assume $\Lambda_1, \Lambda_2 \in \mathrm{X}_2$ are commensurable. Show that*

1. *$(\mathbf{a}_t\Lambda_1)_{t>0}$ diverges*[1] *iff $(\mathbf{a}_t\Lambda_2)_{t>0}$ diverges;*
2. *$\mathrm{A}.\Lambda_1$ is bounded (i.e., closure is compact) iff $\mathrm{A}.\Lambda_2$ is bounded.*

Exercise I. *For $\varepsilon > 0$, let $B_\varepsilon := \{x \in \mathbb{R}^2,\ \|x\| < \varepsilon\}$. Show that for any $\Lambda \in \mathrm{X}_2$, one has $\Lambda \cap B_1 \subset \mathbb{Z}.\mathbf{v}$ for some $\mathbf{v} \in \Lambda$.*

Exercise J. *For $\alpha \in [0, 1)$, let $\Lambda_\alpha \in \mathrm{X}_2$ be as in the lecture notes. Show that $(\mathbf{a}_t.\Lambda_\alpha)_{t>0}$ diverges iff $\alpha \in \mathbb{Q}$.*

(Hint: you might want to use Exercise I.)

Exercise K. *Use Exercise J to give another proof of the fact that for some constant $C > 0$, for every irrational number α, there are infinitely many $q \in \mathbb{Z}^+$ such that $q\langle q\alpha\rangle < C$.*

Below we sketch, in the form of exercises, how to prove an inhomogeneous analogue of this.

Definition 0.2. *We define*

$$\mathrm{Y}_2 := \left\{ (\Lambda, \mathbf{v} + \Lambda) \,\middle|\, \Lambda \in \mathrm{X}_2,\ \mathbf{v} + \Lambda \in \mathbb{R}^2/\Lambda \right\}$$

An element $(\Lambda, \mathbf{v} + \Lambda) \in \mathrm{Y}_2$[2] *is referred to as a **unimodular grid**. A sequence $(\Lambda_n, \mathbf{v}_n + \Lambda_n)$ converges to $(\Lambda, \mathbf{v} + \Lambda)$ iff there are $\mathbf{x}_n, \mathbf{y}_n, \mathbf{v}'_n \in \mathbb{R}^2$ and $\mathbf{x}, \mathbf{y}, \mathbf{v}' \in \mathbb{R}^2$, such that*

$$\Lambda_n = \mathbb{Z}\mathbf{x}_n + \mathbb{Z}\mathbf{y}_n,\ \mathbf{v}_n + \Lambda_n = \mathbf{v}'_n + \Lambda_n,\ \Lambda = \mathbb{Z}\mathbf{x} + \mathbb{Z}\mathbf{y},\ \mathbf{v} + \Lambda = \mathbf{v}' + \Lambda;$$
$$(\mathbf{x}_n) \text{ converges to } \mathbf{x},\ (\mathbf{y}_n) \text{ converges to } \mathbf{y},\ \text{and } (\mathbf{v}'_n) \text{ converges to } \mathbf{v}.$$

Also note that $\mathbf{SL}_2(\mathbb{R})$ acts on Y_2 by $(g, \mathbf{v} + \Lambda) \mapsto g\mathbf{v} + g\Lambda$.

Exercise L. *Let B_ε be as above. Show that for $\varepsilon > 0$ sufficiently small (say, $\varepsilon = 0.01$ should suffice), for any unimodular grid $(\Lambda, \mathbf{v} + \Lambda)$, one has that $B_\varepsilon \cap (\mathbf{v} + \Lambda)$ is contained in a line (not necessarily passing through the origin).*

For $\alpha, \beta \in [0, 1)$, define a unimodular grid by $y_{\alpha,\beta} = (\Lambda, (\beta, 0)^{\mathrm{tr}} + \Lambda) \in \mathrm{Y}_2$[3].

Exercise M. *Take $\alpha, \beta \in [0, 1)$. The following two are equivalent:*

1. *for any $\varepsilon > 0$, there exists $t_0 > 0$ such that for all $t > t_0$, $\mathbf{a}_t.y_{\alpha,\beta} \cap B_\varepsilon \neq \emptyset$;*
2. *$\beta \in \mathbb{Z} + \mathbb{Z}\alpha$.*

Exercise N. *Using the above two exercises (or use any other methods you might know) to show that for some constant $C > 0$, for every $\alpha, \beta \in [0, 1)$ with $\beta \notin \mathbb{Z} + \mathbb{Z}\alpha$, there are infinitely many $q \in \mathbb{Z}^+$, such that $q\langle q\alpha + \beta\rangle < C$.*

Recall that in the first lecture, the homogeneous version was deduced from a theorem of Dirichlet, which is no longer true in the inhomogeneous setting.

Exercise O. *Prove that for any $c > 0$, there exist $\alpha, \beta \in \mathbb{R}$ such that there exists infinitely many $N \in \mathbb{Z}^+$ such that for every $q \in \{0, 1, ..., N - 1\}$,*

$$\langle q\alpha + \beta\rangle > \frac{c}{N}.$$

Actually, maybe your proof is good enough to show the same conclusion holds replacing $\forall c > 0$ and $\frac{c}{N}$ by any other function $N \mapsto \psi(N)$ decreasing to 0 (the choice of α, β would depend on this ψ).

Below we give an application that does not belong to Diophantine approximation.

[1] i.e., for any compact subset $\mathscr{C} \subset \mathrm{X}_2$, there exists $t_0 > 0$ such that for all $t > t_0$, $\mathbf{a}_t\Lambda_1 \notin \mathscr{C}$.

[2] this is sometimes abbreviated as $\mathbf{v} + \Lambda$

[3] we take transpose of a row vector as by convention, we write vectors as column vectors

Exercise P. *Show that there exists a bounded set in* X_2 *such that every* A*-orbit intersects with that bounded set non-trivially.*

(hint: you might want to use Exercise I).

Let

$$X(\mathbb{R}) := \{M \in \mathbf{Mat}_{2\times 2}(\mathbb{R}) \mid \det(M) = -1,\ \mathrm{Trace}(M) = 0\}$$
$$X(\mathbb{Z}) := \{M \in \mathbf{Mat}_{2\times 2}(\mathbb{Z}) \mid \det(M) = -1,\ \mathrm{Trace}(M) = 0\}$$

Note that $\mathbf{SL}_2(\mathbb{R})$ acts on $X(\mathbb{R})$ by $(g, M) \mapsto gMg^{-1}$. Similarly, $\mathbf{SL}_2(\mathbb{Z})$ acts on $X(\mathbb{Z})$.

Exercise Q. *Show that the action* $\mathbf{SL}_2(\mathbb{R}) \curvearrowright X(\mathbb{R})$ *is transitive.*

Exercise R. *Show that the action* $\mathbf{SL}_2(\mathbb{Z}) \curvearrowright X(\mathbb{Z})$ *has only finitely many orbits.*

(Hint: note that the stabilizer of the diagonal matrix $\mathrm{diag}(1, -1)$ in $\mathbf{SL}_2(\mathbb{R})$ contains A. Then use Exercise P. Of course you are welcome to use any other method.)

Essentially, Lemma 1.23 from the Lecture 2 shows that the map

$$\mathbf{SL}_2(\mathbb{R}) \to \mathcal{U} := \text{Unitary Operators on } L^2(\mathrm{X}_2, \mathrm{m}_{\mathrm{X}_2})$$

defined by $g \mapsto U_g$ with $(U_g.\phi)(x) := \phi(g^{-1}x)$ is continuous if $\mathcal{U}$ is equipped with the "strong operator topology".

Exercise S. *Show that this map form* $\mathbf{SL}_2(\mathbb{R}) \to \mathcal{U}$ *is not continuous if* $\mathcal{U}$ *is equipped with the operator norm topology.*

The purpose of the following two exercises is to show you a curious calculation.

Exercise T. *Define*

$$\mathcal{R} := \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; -\frac{1}{2} < x < \frac{1}{2},\ x^2 + y^2 > 1,\ y > 0\right\}$$

Calculate the following double integral

$$\int_{\mathcal{R}} \frac{\mathrm{dxdy}}{y^2} = \frac{\pi}{3}.$$

Exercise U. *Let* p *be a prime number, show that* $\#\,\mathbf{SL}_2(\mathbb{Z}/p\mathbb{Z}) = (p^2 - 1)p$.

(Hint: find out $\#\,\mathbf{GL}_2(\mathbb{Z}/p\mathbb{Z})$ first.)

Remark. It is not necessary to read this, but here are some contexts about the exercises T and U above. Exercise T shows that with respect to the volume form as defined by "$\frac{\mathrm{dxdydz}}{|x|}$", any (strict) fundamental domain for $\mathbf{SL}_2(\mathbb{Z})$ has volume $\frac{\pi}{3} \times \frac{\pi}{2} = \frac{\pi^2}{6}$. With respect to the same volume form, one can show that the volume of $\mathbf{SL}_2(\mathbb{Z}_p)$ (p-adic integers) is equal to $p^{-3}\,|\mathbf{SL}_2(\mathbb{Z}/p\mathbb{Z})|$ (which is equal to $1 - p^{-2}$ by Exercise U). The fact that $\zeta(2) = \frac{\pi^2}{6}$ shows that

$$\left(\frac{\pi}{3} \times \frac{\pi}{2}\right) \cdot \prod \frac{(p^2 - 1)p}{p^3} = \frac{\pi^2}{6} \cdot \prod (1 - p^{-2}) = 1.$$

By putting things together we have

$$\mathrm{Vol}(\mathbf{SL}_2(\mathbb{R}) / \mathbf{SL}_2(\mathbb{Z})) \times \prod_p \mathrm{Vol}(\mathbf{SL}_2(\mathbb{Z}_p)) = 1.$$

In adelic language, this can be restated as $\mathrm{Vol}(\mathbf{SL}_2(\mathbb{A}_{\mathbb{Q}}) / \mathbf{SL}_2(\mathbb{Q})) = 1$. $\square$

Hopefully the next three exercises can help you understand Lemma 1.26 from "Lecture 2" better. For this purpose, let $\pi : \mathbf{SL}_2(\mathbb{R}) \to \mathbf{SL}_2(\mathbb{R}) / \mathbf{SL}_2(\mathbb{Z})$ be the natural quotient map. Recall that it is open and continuous. For $x \in \mathbf{SL}_2(\mathbb{R}) / \mathbf{SL}_2(\mathbb{Z})$, define $\mathrm{Obt}_x : \mathbf{SL}_2(\mathbb{R}) \to \mathbf{SL}_2(\mathbb{R}) / \mathbf{SL}_2(\mathbb{Z})$ by $\mathrm{Obt}_x(g) := g.x$. Then Obt_x is also open and continuous.

For our convenience, fix a distance function (i.e., a metric) $d(\cdot, \cdot) : \mathbf{SL}_2(\mathbb{R}) \times \mathbf{SL}_2(\mathbb{R}) \to \mathbb{R}_{\geq} 0$[4]. For $\varepsilon > 0$, let $B_{\mathrm{I}_2}(\varepsilon) := \{g \in \mathbf{SL}_2(\mathbb{R}),\ d(g, \mathrm{I}_2) < \varepsilon\}$ (recall I_2 denotes the two-by-two identity matrix).

[4]you can choose any metric you like as long as it is compatible with the topology.

Exercise V. *Show that for any compact subset $\mathscr{C} \subset \mathbf{SL}_2(\mathbb{R})$, there exists $\varepsilon > 0$ such that for every $h \in \mathscr{C}$, the map*

$$\mathrm{Obt}_{\pi(h)} : B_{\mathrm{I}_2}(\varepsilon) \to B_{\mathrm{I}_2}(\varepsilon).\pi(h)$$

is bijective.

(Hint: it boils down to showing that for some $\eta > 0$, for every $h \in \mathscr{C}$ and $\gamma_{\neq \mathrm{I}_2} \in \mathbf{SL}_2(\mathbb{Z})$, $d(h\gamma h^{-1}, \mathrm{I}_2) > \eta$.

Bijection + openness + continuity imply that $\mathrm{Obt}_{\pi(h)} : B_{I_2}(\varepsilon) \to B_{I_2}(\varepsilon).\pi(h)$ is a homeomorphism.

For $\varepsilon > 0$, define $\mathrm{Cor}_\varepsilon : (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \to \mathbf{SL}_2(\mathbb{R})$ by

$$(t, s, r) \mapsto \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} = \mathbf{a}_t \cdot \mathbf{u}_s^- \cdot \mathbf{u}_r^+$$

Exercise W. *Show that for $\varepsilon > 0$ small enough,*

1. $\mathrm{Image}(\mathrm{Cor}_\varepsilon)$ *is open and that* $\mathrm{Cor}_\varepsilon : (-\varepsilon, \varepsilon)^3 \to \mathrm{Image}(\mathrm{Cor}_\varepsilon)$ *is a homeomorphism.*

Moreover, there exists some positive continuous function $\lambda_\varepsilon : (-\varepsilon, \varepsilon)^3 \to \mathbb{R}_{>0}$ such that

2. $(\mathrm{Cor}_\varepsilon)_*(\lambda_\varepsilon \cdot |\mathrm{dtdsdr}|)$ *is equal to the restriction of* $\mathrm{m}_{\mathbf{SL}_2(\mathbb{R})}$ *to* $\mathrm{Image}(\mathrm{Cor}_\varepsilon)$.

(Hint: you might want to use the open set (actually a neighborhood of I_2) $\mathcal{O}_1 := \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \mathbf{SL}_2(\mathbb{R}) \,\middle|\, x \neq 0 \right\}$ and the local coordinate

$$\mathcal{O}'_1 := \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, x \neq 0 \right\} \xrightarrow[\simeq]{\varphi_1} \mathcal{O}_1$$

with $\varphi_1(x, y, z) := \begin{bmatrix} x & y \\ z & \frac{1+yz}{x} \end{bmatrix}$ in the explicit construction of $\mathrm{m}_{\mathbf{SL}_2(\mathbb{R})}$. Under this coordinate, $\mathrm{m}_{\mathbf{SL}_2(\mathbb{R})}$ was defined by $(\varphi_1)_*(\frac{|\mathrm{dxdydz}|}{|x|})$.)

Exercise X. *Combine the efforts from the above two exercises to prove that for every $\mathscr{C} \subset \mathbf{SL}_2(\mathbb{R}) / \mathbf{SL}_2(\mathbb{Z})$ compact, there exists $\varepsilon > 0$ such that for every $\Lambda \in \mathbf{SL}_2(\mathbb{R}) / \mathbf{SL}_2(\mathbb{Z})$, there exists a continuous positive function $\lambda_{\Lambda, \varepsilon} : (-\varepsilon, \varepsilon)^3 \to \mathbb{R}_{>0}$ such that*

$$\left(\mathrm{Obt}_\Lambda \circ \mathrm{Cor}_\varepsilon\right)_* \left(\lambda_{\Lambda,\varepsilon}(t, s, r)\, |\mathrm{dtdsdr}|\right)$$

is equal to the restriction of $\mathrm{m}_{\mathrm{X}_2}$[5] *to the image of* $\mathrm{Obt}_\Lambda \circ \mathrm{Cor}_\varepsilon$.

[5] When writing $\mathrm{m}_{\mathrm{X}_2}$, we have (implicitly) identified X_2 with $\mathbf{SL}_2(\mathbb{R}) / \mathbf{SL}_2(\mathbb{Z})$.